Solutions for HW 1 285s9

Note Title 2/20/2009

Section 1.1: 9, 13, 19, 24, 33, 35
Section 1: 3, 9, 18

Problem 1.1, 9: Verify that $y(x) = \dfrac{1}{1+x^2}$ is a solution of

$$y' + 2xy^2 = 0,$$

Solution: Calculate $y'(x) = \dfrac{-1}{(1+x^2)^2} \cdot 2x = \dfrac{-2x}{(1+x^2)^2}$,

hence

$$y' + 2xy^2 = \frac{-2x}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2} = 0 \qquad \square$$

Problem 1.1; 13: Substitute e^{rx} into $3y' = 2y$ and determine the value of r for which e^{rx} is a solution.

Solution: If $y(x) = e^{rx}$, then $y'(x) = re^{rx}$. Thus $3y' = 2y$ yields

$$3re^{rx} = 2e^{rx}$$

$$\Rightarrow \quad 3r = 2 \quad \text{or} \quad r = \frac{2}{3},$$

i.e., $y(x) = e^{\frac{2}{3}x}$ is a solution of $3y' = 2y$. $\square$

Problem 1.1; 19 and 24: Verify that the given function y solves the ODE and then determine the value of C s.t. y satisfies the given initial condition.

19: $y' = y + 1$, $y(x) = Ce^x - 1$, $y(0) = 5$.

Solution: $y'(x) = Ce^x = Ce^x - 1 + 1 = y(x) - 1$.

so the given fct y solves the ODE.

If $5 = y(0) = C-1 \Rightarrow C=6$

So $y(x) = 6e^{x}-1$ satisfies the given initial cond.

24: $xy' - 3y = x^3$, $y(x) = x^3(C+\ln x)$, $y(1) = 17$

Solution: $y'(x) = 3x^2(C+\ln x) + x^3 \frac{1}{x}$
$= 3x^2(C+\ln x) + x^2$

and $xy - 3y = x(3x^2(C+\ln x)+x^2) - 3x^3(C+\ln x)$
$= x^3$.

Initial cond:
$17 = y(1) = 1^3(C+\ln 1) = C$

so $C = 17$,

$\Rightarrow$ $y(x) = x^3(17+\ln x)$ solves the ODE
and satisfies the initial cond.

Problem 1.1; 33: The rate of change in the velocity v of a coasting motorboat is proportional to the square of v.

Solution: $\frac{dv}{dt} = -\alpha v^2$ $(\alpha > 0)$

This is a separable equation, so we can solve it:

Rewrite $\frac{dv}{dt} = -\alpha v^2$ as $\frac{dv}{v^2} = -\alpha\, dt$

and integrate $\displaystyle\int \frac{dv}{v^2} = -\alpha \int dt + C$

$$\Rightarrow \quad -v^{-1} = C - \alpha t$$

$$\Rightarrow \quad v = \frac{1}{\alpha t - C}. \qquad \square$$

Problem 1.1; 35: In a city of P persons (fixed),

the time rate of change of the number of persons N who have heard a certain rumor is proportional to the number of those who have not heard the rumor. Find an ODE for N.

Solution: total population $= P$
people knowing the rumor $= N$
people not knowing the rumor $= P - N$

$$\Rightarrow \quad \frac{dN}{dt} = \alpha (P-N). \qquad \square$$

Problem 1.2; 3 & 9: Find a solution of the given ODE

and the prescribed initial condition.

3: $\dfrac{dy}{dx} = \sqrt{x}, \quad y(4) = 0.$

Solution: Just integrate both sides

$$y(x) - \underset{\parallel}{y(4)} = \int_4^x \sqrt{s}\, ds = \left[\tfrac{2}{3} s^{3/2} \right]_4^x$$

$$= \frac{2}{3}\left(x^{3/2} - 4^{3/2}\right)$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{3}2^3$$

$$\Rightarrow y(x) = y(4) + \frac{2}{3}x^{3/2} - \frac{2}{3}2^3$$

$$= \frac{2}{3}x^{3/2} - \frac{16}{3}. \qquad \boxed{\;}$$

9: $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$, $y(0) = 0$.

Sol: The antiderivative of $\dfrac{1}{\sqrt{1-x^2}}$ is $\underbrace{\sin^{-1}(x)}_{=\arcsin(x)} + C$

Thus integrating both sides gives

$$y(x) = \int \frac{1}{\sqrt{1-x^2}}\,dx + C = \arcsin(x) + C$$

Since $0 = y(0) = \arcsin(0) + C = C$

$$\Rightarrow y(x) = \arcsin(x). \qquad \boxed{\;}$$

Problem 1.2; 18: Find the position of a particle with acceleration $a(t) = 50\sin 5t$, $v_0 = -10$, $x_0 = 8$

Solution

$$x''(t) = a(t) = 50\sin 5t$$

$$\Longrightarrow \quad x'(t) = \int_0^t 50 \sin 5s \, ds + x'(0)$$

integrate once

$$= \Big[-10 \cos 5s\Big]_0^t + v_0$$

$$= -10 \cos 5t + 10 + v_0$$

and

$$x(t) = \int_0^t x'(s)\, ds + x(0)$$

$$= \int_0^t (-10 \cos 5s + v_0 + 10)\, dt + x_0$$

$$= \Big[2 \sin 5s \Big]_0^t + (v_0 + 10)t + x_0$$

$$= 2 \sin 5t + (v_0 + 10)t + x_0$$

Since $v_0 = -10$ and $x_0 = 8$, we get

$$\boxed{x(t) = 2 \sin 5t + 8.}$$

$\square$