Solutions HW 10 285s09

Section 9.1: 13, 17

9.2: 3, 18

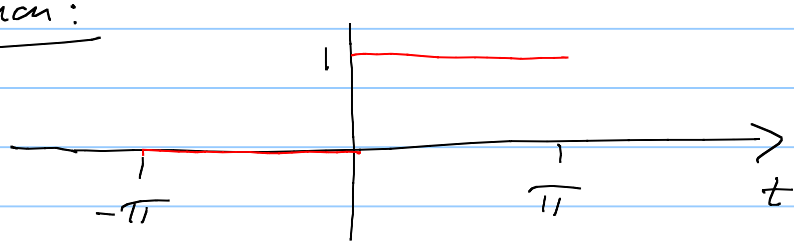
9.4: 3, 7, 11

9.1.13: The 2π periodic fct. $f(t)$ is given by

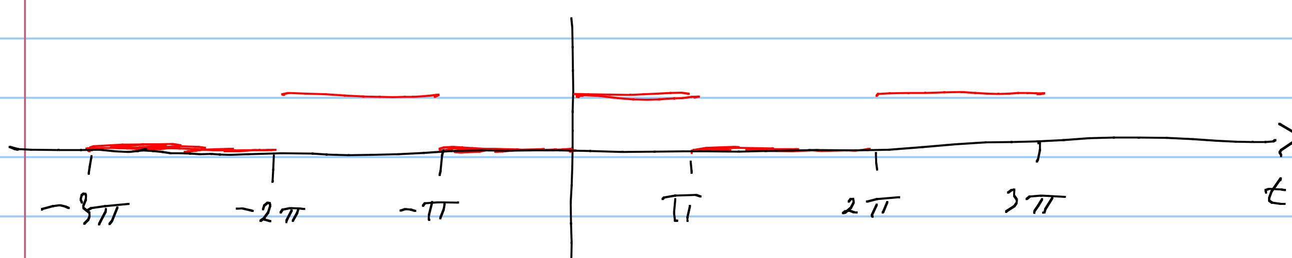
$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

Sketch f and find the Fourier series of f

Solution:



$\Rightarrow$ over several periods



Fourier series:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos(nt)\,dt$$

$n=0$: $$a_0 = \frac{1}{\pi}\int_0^{\pi} dt = 1$$

$n \geq 1$: $$a_n = \frac{1}{\pi}\int_0^{\pi} \cos(nt)\,dt = \frac{1}{\pi n}\int_0^{n\pi} \cos(r)\,dr$$

$$= \frac{1}{\pi n}\Big[\sin r\Big]_0^{n\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin(nt)\,dt = \frac{1}{n\pi}\int_0^{n\pi}\sin r\,dr$$

$$= \frac{1}{n\pi}\Big[\cos r\Big]_0^{n\pi} = \frac{1}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= \frac{1}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \dfrac{-2}{n\pi} & n \text{ odd} \end{cases}$$

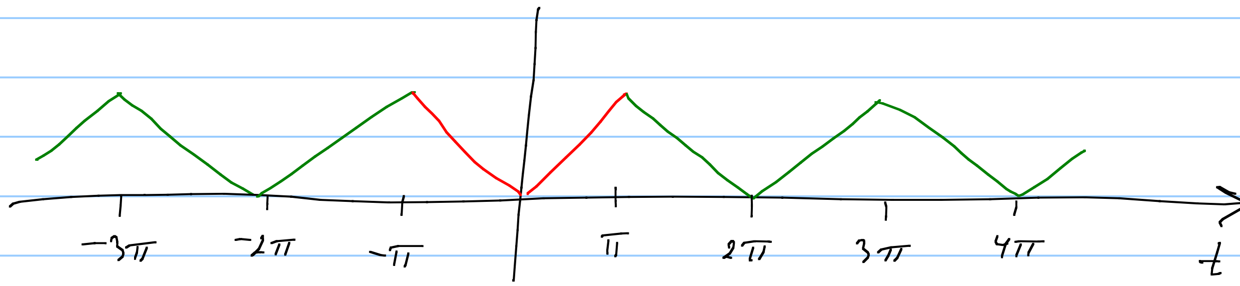
So the Fourier series of f is given by

$$\frac{1}{2} - \frac{2}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}$$

9.1.17: Same as above for the 2π periodic fcn. given by

$$f(t) = |t| \quad , -\pi < t < \pi$$

Solution: The graph of f is given by



Fourier series:

f is even so $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \sin(nt)\,dt = 0$

and $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \cos(nt)\,dt = \frac{2}{\pi} \int_0^{\pi} t \cos(nt)\,dt$

$n=0$: $a_0 = \frac{2}{\pi} \int_0^{\pi} t\,dt = \frac{2}{\pi} \left[\frac{t^2}{2} \right]_0^{\pi} = \pi$

$n \geq 1$: $a_n = \frac{2}{\pi} \int_0^{\pi} t \cos(nt)\,dt$ subst. $r = nt$, $dt = \frac{dr}{n}$

$$= \frac{2}{\pi n^2} \int_0^{n\pi} \underset{u}{r} \underset{v'}{\cos r}\,dr$$

$$= \frac{2}{\pi n^2} \left(\left[r \sin r \right]_0^{n\pi} - \int_0^{n\pi} \sin r\,dr \right)$$

$$= \frac{2}{\pi n^2}\left(\left[r \sin r \right]_0^{n\pi} + \left[\cos r \right]_0^{n\pi} \right)$$

$$= \frac{2}{\pi n^2}\Big(\underbrace{\cos(n\pi)}_{(-1)^n} - 1 \Big) = \begin{cases} 0 & n \text{ even} \\ \dfrac{-4}{\pi n^2} & n \text{ odd} \end{cases}$$

So the Fourier series for f is

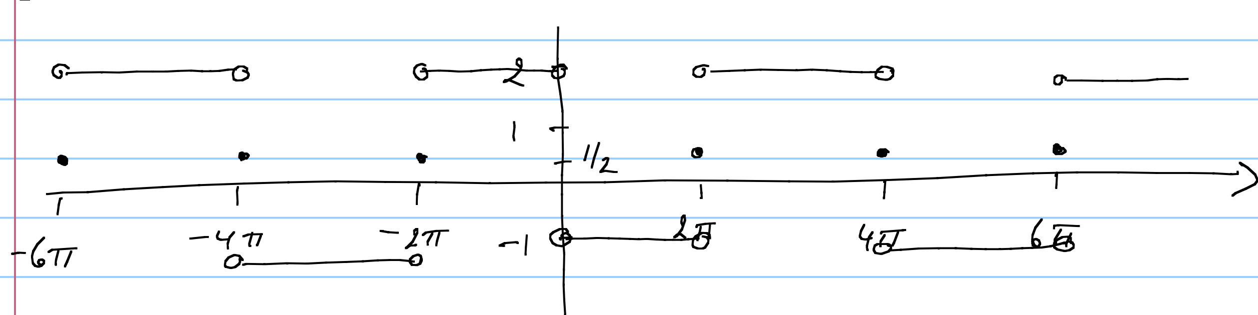
$$\boxed{\frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n^2}}$$

Q 2.3: On one period f is given by

$$f(t) = \begin{cases} 2 & -2\pi < t < 0 \\ -1 & 0 < t < 2\pi \end{cases}$$

Find the Fourier transform and sketch f.

Solution: Sketch of f:



Fourier series : periodic 4π , so $L = 2\pi$

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(t)\cos\left(\frac{n\pi t}{2\pi}\right)dt$$

$$= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(t)\cos\left(\frac{nt}{2}\right)dt$$

$n=0$: $$a_0 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(t)\,dt$$

$$= \frac{1}{2\pi}\left\{\int_{-2\pi}^{0} -1\,dt + \int_{0}^{2\pi} 2\,dt\right\} = \frac{1}{2\pi}\left\{-2\pi + 4\pi\right\}$$

$$= 1$$

$n \geq 1$: $$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(t)\cos\left(\frac{nt}{2}\right)dt$$

$$= \frac{1}{2\pi}\left\{\int_{-2\pi}^{0}(-1)\cos\left(\frac{nt}{2}\right)dt + \int_{0}^{2\pi} 2\cos\left(\frac{nt}{2}\right)dt\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{2}{n}\int_{-n\pi}^{0}\cos r\,dr + \frac{4}{n}\int_{0}^{n\pi}\cos r\,dr\right\}$$

$r = \frac{nt}{2}$

$$= \frac{1}{2\pi}\,\frac{2}{n}\int_{0}^{n\pi}\cos r\,dr = \frac{1}{n\pi}\left[-\sin r\right]_0^{n\pi} = 0$$

symmetry of cos.

$$b_n = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(t) \sin\left(\frac{nt}{2}\right) dt$$

$$= \frac{1}{2\pi} \left\{ \int_{-2\pi}^{0} -\sin\left(\frac{nt}{2}\right) dt + \int_{0}^{2\pi} 2\sin\left(\frac{nt}{2}\right) dt \right\}$$

$r = \frac{nt}{2}$

$$= \frac{1}{n\pi} \left\{ \underbrace{-\int_{-n\pi}^{0} \sin(r)\, dr}_{= \int_0^{n\pi} \sin(r)\, dr} + 2 \int_{0}^{2\pi} \sin(r)\, dr \right\}$$

$$= \frac{3}{n\pi} \int_0^{n\pi} \sin r\, dr = \frac{3}{n\pi} \left[\cos r\right]_0^{n\pi} = \frac{3}{n\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{-6}{n\pi} & n \text{ odd} \end{cases}$$

$\Rightarrow$ Fourier series of f is

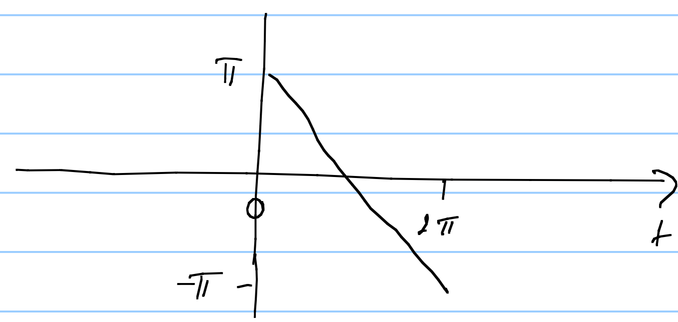
$$f(t) = \frac{1}{2} - \frac{6}{\pi} \sum_{n \text{ odd}} \frac{\sin\left(\frac{nt}{2}\right)}{n} .$$

9.2;18: Given is $\sum_{n=1}^{\infty} \frac{\sin nt}{n} = \frac{\pi - t}{2}$ (for $0 < t < 2\pi$)

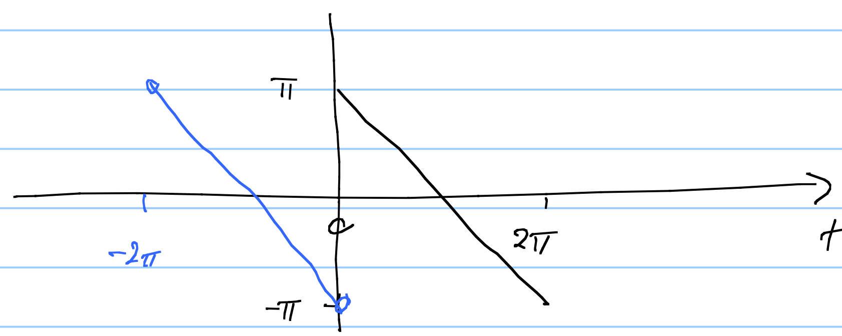
derive this Fourier series and sketch the function to which it converges

Solution: We have a Fourier sine series here, so f must be odd! On $0 < t < 2\pi$ f is given by

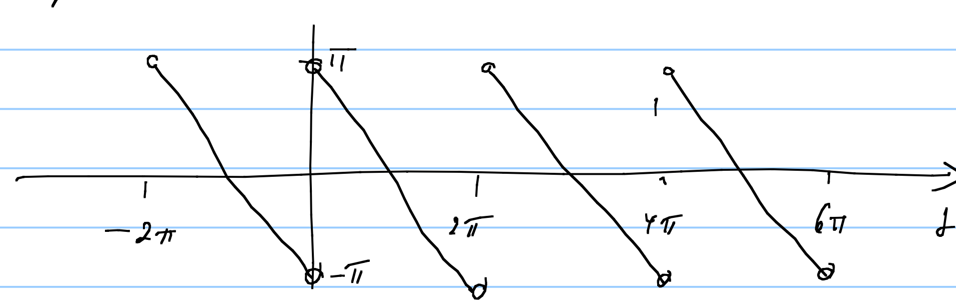
$$f(t) = \frac{\pi - t}{2}$$



Since it is odd, we must extend it to $(-2\pi, 0)$ as



$\Rightarrow$ the graph of f looks like



and f is a 2π-periodic function!

2 ways to calculate the Fourier series

over one period $[0, 2\pi]$ or on the period $[-2\pi, 2\pi]$

on $[-2\pi, 2\pi]$ f is odd, so all the corresponding a_n must vanish.

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt)\, dt$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi - t}{2} \sin(nt)\, dt$$

$$= \underbrace{\int_0^{2\pi} \sin(nt)\, dt}_{=0} - \frac{1}{2\pi} \int_0^{2\pi} t \sin(nt)\, dt$$

$$= \dots = \frac{1}{n}$$

same calculation ....

9.4; 3: $x'' + 3x = F(t)$, F odd 2π periodic with $F(t) = 2t \quad 0 < t < \pi$

Find a part. sol. x_{sp}

Solution: The odd extension of F to $(-\pi, \pi)$ is

$$F(t) = 2t \quad \text{for } -\pi < t < \pi$$

F has Fourier series

$$F(t) = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)$$

The solution of the homog. eq. $x'' + 3x = 0$ is given by

$$x_c(t) = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)$$

Since $nt \neq \sqrt{3}t$ for all $n \geq 1$ and t there is no resonance in any term of the Four series of F

$\Rightarrow$ guess for

$$x_{sp}(t) = \sum_{n=1}^{\infty} c_n \sin(nt)$$

$$\Rightarrow x_{sp}'' + 3x_{sp} = \sum_{n\geq 1} c_n(-n^2+3)\sin(nt)$$

$$\stackrel{!}{=} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)$$

Comparing coefficients $\Rightarrow$

$$c_n = \frac{(-1)^{n+1}}{n(3-n^2)} \quad \text{so}$$

$$x_{sp}(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(3-n^2)} \sin(nt)$$ ∎

9.4.7: Let $mx'' + kx = F(t)$

F odd 2π periodic fct with $F(t) = 1$ $0 < t < \pi$.

If $m=1$, $k=9$ is there a resonance?

Solution:

The circular frequency of the free motion

$mx''+kx=0$ is given by $\omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{9}{1}}=3$

The Fourier sine series of F is given by

$$F(t)=\underbrace{\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}}$$

contains the term $\frac{1}{3}\sin(3t)$

So a resonance occurs!

9.4.11: As above, but with $m=3$, $k=48$

F is the even 2π periodic fct with $F(t)=t$, $0<t<\pi$

Solution: circular frequency of $mx''+kx=0$

is $\omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{48}{3}}=\sqrt{16}=4$

so the general solution of the homog. eq.

$3x''+48x=0$ is given by

$$x_c = C_1 \cos(4t) + C_2 \sin(4t)$$

The Fourier cosine series of F is (after some calculation)

$$F(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

A resonance would occur if this series contains a term of the complementary solution $x_c = C_1 \cos(4t) + C_2 \sin(4t)$

This would be the $n = 4$ term in the Fourier cosine series for F.

However, the series for F is only over n odd

so it contains no $\cos(4t)$ term

$\Rightarrow$ NO resonance!

$\boxed{\text{M}}$