Solutions for HW 9 285 s09

Section 3.5 : 58
Section 3.6 : 15, 18

58: Find a particular solution of

$$x^2y'' - 4xy' + 6y = x^3$$

given that the general solution of the homog. equation

$$x^2y'' - 4xy' + 6y = 0$$

is given by $y_c = c_1x^2 + c_2x^3$

Solution: We will use variation of parameters method.

Rewrite the ODE in standard form:

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x \qquad (*)$$

We search for a solution in the form

$$y_p = u_1x^2 + u_2x^3 \qquad , u_1, u_2 \text{ functions of } x$$

$$y_p' = u_1'x^2 + u_2'x^3 + 2u_1x + 3u_2x^2$$

Assume that $\boxed{u_1'x^2 + u_2'x^3 = 0}$, then

$$y_p' = 2u_1x + 3u_2x^2$$

and $y_p'' = 2u_1'x + 2u_1 + 3u_2'x^2 + 6u_2x$

Plug this into $(*)$, to get

$$x = y_p'' - \frac{4}{x}y_p' + \frac{6}{x^2}y_p$$

$$= 2u_1'x + 2u_1 + 3u_2'x^2 + 6u_2x$$

$$- \frac{4}{x}(2u_1x + 3u_2x^2) + \frac{6}{x^2}(u_1x^2 + u_2x^3)$$

$$= 2u_1'x + 3u_2'x^2 + 2u_1 - 4u_1 + 6u_1 = 0$$

$$+ 6u_2x - 12u_2x + 6u_2x = 0$$

$$= 2u_1'x + 3u_2'x^2$$

So we have the 2 equations

$$\text{I}: \quad x^2u_1' + x^3u_2' = 0$$

$$\text{II}: \quad 2xu_1' + 3x^2u_2' = x$$

$$2\cdot\text{I} - x\,\text{II} = 2x^2u_1' + 2x^3u_2' - 2x^2u_1 - 3x^3u_2' = -x^2$$

$$\Rightarrow \quad x^3u_2' = x^2 \quad \Rightarrow \quad \boxed{u_2' = \frac{1}{x}}$$

and from I $\Rightarrow$ $\boxed{u_1' = -xu_2' = -1}$

Hence $u_1 = -\int dx = -x$

$$u_2 = \int \frac{dx}{x} = \ln x$$

So a particular solution is given by

$$\boxed{y_p = u_1 x^2 + u_2 x^3 = -x^3 + x^3 \ln x}$$

3.6.15: See whether there is a practical resonance in the system
& 18

$$mx'' + cx' + kx = F_0 \cos \omega t$$

with

15: $m = 1,\ c = 2,\ k = 2,\ F_0 = 2$

Solution: Well one reason why I don't like the book is that the answer does not depend on F_0 !

We know that the amplitude $C(\omega)$ is given by

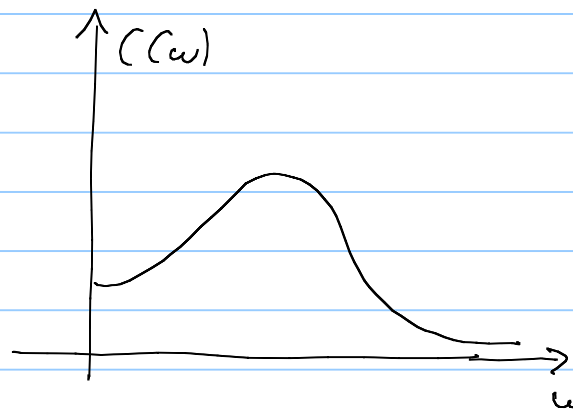
$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

So using $m = 1$, $c = 2$, $k = 2$ yields

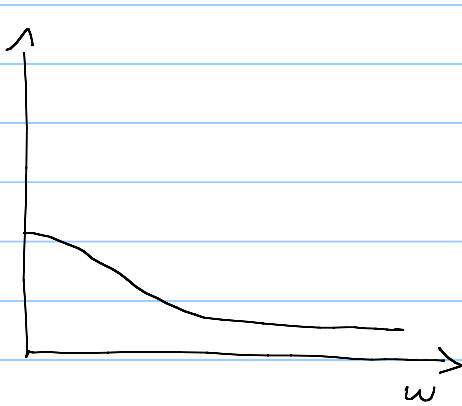
$$C(\omega) = \frac{F_0}{\sqrt{(2-\omega^2)^2 + (2\omega)^2}}$$

$$= F_0 \left((2-\omega^2)^2 + 4\omega^2 \right)^{-1/2}$$

The graph of $\omega \mapsto C(\omega)$ can look either like



or



In the first case $C'(\omega)$ is positive for small ω and negative later, in the second case $C'(\omega)$ is negative for all $\omega > 0$.

$\Rightarrow$ need to calculate $C'(\omega)$

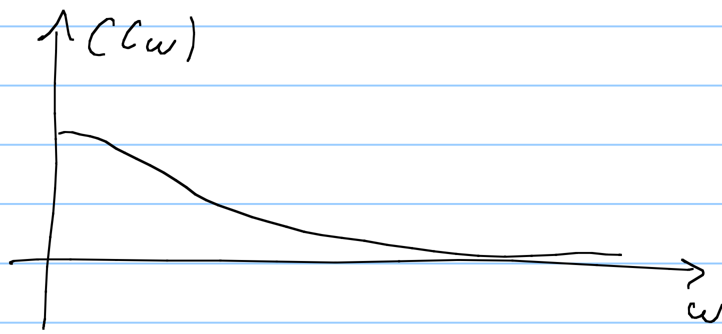
$$C'(\omega) = -\frac{F_0}{2} \left((2-\omega^2)^2 + 4\omega^2 \right)^{-3/2} \left(2(2-\omega^2)(-2\omega) + 8\omega \right)$$

$$= -\frac{F_0 \omega \quad \omega^2}{\left((2-\omega^2)^2 + 4\omega^2 \right)^{3/2}}$$

Since $\omega^2 > 0$ and $F_0 = 2 > 0$, we must have

$C'(\omega) < 0$ for all $\omega > 0$

$\Rightarrow$ the graph of $C(\omega)$ looks like



So there is no practical resonance!

18: here $m = 1$, $c = 10$, $k = 650$, $F_0 = 100$

$$\text{So } C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$= F_0\left((650 - \omega^2)^2 + (10\omega)^2\right)^{-1/2}$$

$$= F_0\left((650 - \omega^2)^2 + 100\omega^2\right)^{-1/2}$$

$$\Rightarrow C'(\omega) = \frac{-F_0}{2}(\ldots)^{-3/2}\left(2(650 - \omega^2)(-2\omega) + 200\omega\right)$$

$$= -F_0\,\omega\,\frac{\omega^2 - 1200}{\left((650 - \omega^2)^2 + 100\omega^2\right)^{3/2}}$$

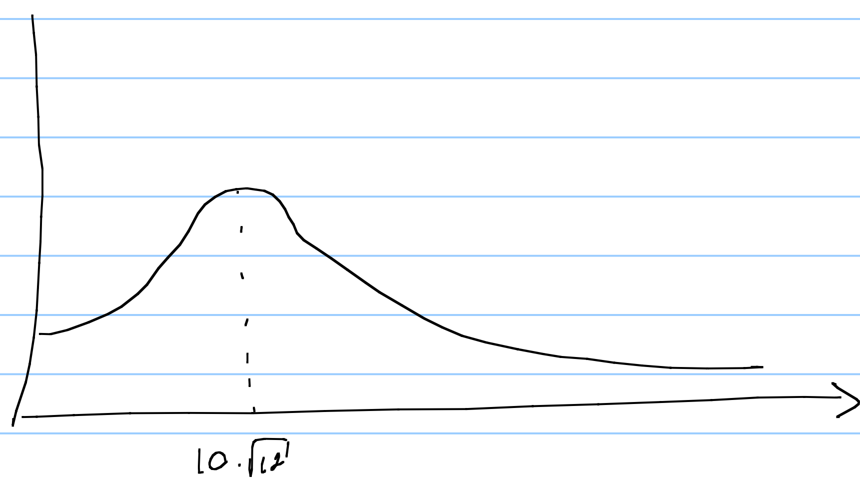
$\Rightarrow C'(\omega) > 0 \iff \omega^2 - 1200 < 0$

$\iff \omega^2 < 1200$

and $C'(\omega) < 0 \iff \omega^2 > 1200$

So $C(\omega)$ has a maximum at $\omega = \sqrt{1200} = \sqrt{12} \cdot 10$

graph



So there is a practical resonance!