IMPROVED HARDY-SOBOLEV INEQUALITIES

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Abstract. The main result includes features of a Hardy-type inequality and an inequality of either Sobolev or Gagliardo-Nirenberg type. It is inspired by the method of proof of a recent improved Sobolev inequality derived by M. Ledoux which brings out the connection between Sobolev embeddings and heat kernel bounds. Here Ledoux's technique is applied to the operator $L := \mathbf{x}\cdot\nabla$ and the analysis requires the determination of the operator semigroup $\{e^{-tL^*L}\}_{t>0}$ and its properties.

1. Introduction

The best possible constant in Hardy's inequality

$$\int_{\mathbb{R}^n} |\nabla f|^p d\mathbf{x} \geq C(n,p) \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^p}{|\mathbf{x}|^p} d\mathbf{x} \tag{1.1}$$

is $C(n,p) = \{(n-p)/p\}^p$ and so the inequality only yields non-trivial information when $n \neq p$. In Theorem 1 below, we prove that the related inequality

$$\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla) f(\mathbf{x})|^p d\mathbf{x} \geq (n/p)^p \int_{\mathbb{R}^n} |f(\mathbf{x})|^p d\mathbf{x} \tag{1.2}$$

is satisfied for all values of n, including $n = p$, and this implies Hardy's inequality for $1 \leq p \leq n$. The case $n = p$ has a special significance also for the Sobolev inequality

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C'(n,p) \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad q = p^* = np/(n-p),\ 1 \leq p < n; \tag{1.3}$$

when $n = p$, (1.3) does not hold for $q = \infty$. In [2], [3] and [7], the following improvement of the Sobolev inequality is derived: for $1 \leq p < q < \infty$,

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C'(n,p) \|\nabla f\|^{p/q}_{L^p(\mathbb{R}^n)} \|f\|^{1-p/q}_{B^{p/(p-q)}_{\infty,\infty}} : \tag{1.4}$$

the space $B^{p/(p-q)}_{\infty,\infty}$ is a Besov space defined in terms of the heat semigroup $e^{t\Delta}$ (c.f.[10], Section 2.5.2). This includes, in particular, the

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Sobolev and Gagliardo-Nirenberg inequalities, and also has important features not possessed by (1.3); see [2], [3] and [7] for details.

This paper has two objectives: first to determine the semigroup e^{-tL^*L}, where $L = \mathbf{x}\cdot\nabla$ in $L^2(\mathbb{R}^n)$, and then to use this to derive an improved version of (1.2) which is analogous to (1.4). A corollary of our main theorem in the case $p = 2$ is the inequality:

$$\begin{aligned} \|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu))} &\le C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \|f\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{1.5}$$

where $2^* = 2n/(n-2), d\mu(r) = r^{n-1}dr, C$ is a positive constant depending only on n and, in polar co-ordinates $\mathbf{x} = r\omega, F(r)$ is the integral mean of f over the unit sphere $\mathbb{S}^{n-1}$, that is,

$$F(r) := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(r\omega)d\omega.$$

This has a number of consequences. One is a Hardy-Sobolev type inequality (Corollary 5) which is that if $f, \nabla f \in L^2(\mathbb{R}^n), n \ge 3$, then,

$$\begin{aligned} \|F(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} &\le C\left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \frac{(n-2)^2}{4}\|f/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \|f/|\cdot|\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)} \end{aligned}$$

which yields, for any $\delta \in [0, (n-2)^2/4)$,

$$\|F\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C[\frac{(n-2)^2}{4} - \delta]^{-\frac{(n-1)}{n}}\left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \delta\|f/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\right\}. \tag{1.6}$$

Since $\|F\|_{L^{2^*}(\mathbb{R}^+;d\mu)} \le |\mathbb{S}^{n-1}|^{-1/2^*}\|f\|_{L^{2^*}(\mathbb{R}^n)}$, by Hölder's inequality, (1.6) is implied by Stubbe's result in [9], Theorem 1, namely

$$\|f\|^2_{L^{2^*}(\mathbb{R}^n)} \le K(n)[\frac{(n-2)^2}{4} - \delta]^{-\frac{(n-1)}{n}}\left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \delta\|\frac{f}{|\cdot|}\|^2_{L^2(\mathbb{R}^n)}\right\} \tag{1.7}$$

with optimal constant

$$K(n) = [\pi n(n-2)]^{-1}\left(\Gamma(n)/\Gamma(n/2)\right)^{2/n}[(n-2)^2/4]^{(n-1)/n}. \tag{1.8}$$

We also establish the following local Hardy-Sobolev type inequalities (see Corollaries 6 and 7): if f is supported in the annulus $A(1/R, R) := \{\mathbf{x} \in \mathbb{R}^n : 1/R \le |\mathbf{x}| \le R\}$ then

$$\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C(\ln R)^{2(n-1)/n}\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - (n^2/4)\|f\|^2_{L^2(\mathbb{R}^n)}\right\}; \tag{1.9}$$

$$\|F\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C(\ln R)^{2(n-1)/n}\left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \left[\frac{n-2}{2}\right]^2\left\|\frac{f}{|\cdot|}\right\|^2_{L^2(\mathbb{R}^n)}\right\}. \tag{1.10}$$

The inequality (1.10) is reminiscent of the case $s = 1$ of (2.6) in [6] (proved in section 6.4); this is also proved in [1]. To be specific, it is that if $f \in C_0^\infty(\Omega)$ and $2 \leq q < 2^*$,

$$\|f\|^2_{L^q(\mathbb{R}^n)} \leq C|\Omega|^{2(1/q-1/2^*)}\left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \left[\frac{n-2}{2}\right]^2\left\|\frac{f}{|\cdot|}\right\|^2_{L^2(\mathbb{R}^n)}\right\}, \tag{1.11}$$

where $|\Omega|$ denotes the volume of Ω. It is noted in [6], Remark 2.4, that, in contrast to (1.10), the q in (1.11) must be strictly less than the critical Sobolev exponent $2^* = 2n/(n-2)$ if Ω includes the origin.

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2. The Hardy-type inequality (1.2)

Theorem 1. *Let $n \geq 1$ and $1 \leq p < \infty$. Then for all $f \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla)f|^p d\mathbf{x} \geq \left(\frac{n}{p}\right)^p \int_{\mathbb{R}^n} |f|^p d\mathbf{x}. \tag{2.1}$$

Proof. For any differentiable function $V : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{div} V|f|^p d\mathbf{x} &= -p\ Re\int_{\mathbb{R}^n} (V\cdot\nabla f)|f|^{p-2}\overline{f}d\mathbf{x}\\ &\leq p\left(\int_{\mathbb{R}^n}|V\cdot\nabla f|^p d\mathbf{x}\right)^{1/p}\left(\int_{\mathbb{R}^n}|f|^p d\mathbf{x}\right)^{(p-1)/p}\\ &\leq \varepsilon^p\int_{\mathbb{R}^n}|V\cdot\nabla f|^p d\mathbf{x} + (p-1)\varepsilon^{-p/(p-1)}\int_{\mathbb{R}^n}|f|^p d\mathbf{x}\end{aligned} \tag{2.2}$$

for any $\varepsilon > 0$. Now choose $V(\mathbf{x}) = \mathbf{x}$ to get

$$\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla)f|^p d\mathbf{x} \geq K(n,\varepsilon)\int_{\mathbb{R}^n}|f|^p d\mathbf{x}$$

where

$$K(n,\varepsilon) = \varepsilon^{-p}\{n-(p-1)\varepsilon^{-p/(p-1)}\}.$$

This takes its maximum value $(n/p)^p$ when $\varepsilon^{p/(p-1)} = p/n$. This proves the theorem. □

Remark 1. *The inequality (2.1) implies (1.1) for $1 \leq p \leq n$. For we have from*

$$\nabla(|\mathbf{x}|f) = \frac{\mathbf{x}}{|\mathbf{x}|}f + |\mathbf{x}|\nabla f$$

that

$$\begin{aligned}\|\nabla(|\mathbf{x}|f)\|_{L^p(\mathbb{R}^n)} &\geq \||\mathbf{x}||\nabla f|\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)}\\ &\geq \|(\mathbf{x}\cdot\nabla)f\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)}\\ &\geq \left(\frac{n-p}{p}\right)\|f\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

whence (1.1) on replacing $f(\mathbf{x})$ by $f(\mathbf{x})/|\mathbf{x}|$.

3. Calculation of the semigroup e^{-tL^*L}

Theorem 2. *Let $L = \mathbf{x}\cdot\nabla, \mathbf{x} = r\omega, r = |\mathbf{x}|$. Then the semigroup e^{-tL^*L} is given by*

$$(e^{-tL^*L}\psi)(\mathbf{x}) = \frac{e^{-tn^2/4}}{\sqrt{4\pi t}} r^{-n/2} \int_0^\infty e^{-\frac{(\ln r - \ln s)^2}{4t}} s^{-n/2}\psi(s\omega)\, s^{n-1} ds \qquad (3.1)$$

Proof. Before embarking on the proof, some preliminary remarks and results might be helpful. The gist of the proof is that after a change of co-ordinates, L^*L is seen to be related to the Laplacian in $\mathbb{R}$, and this then yields the result. The co-ordinate change is determined by the map $\Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ defined by

$$(\Phi\psi)(s,\omega) := e^{sn/2}\psi(e^s\omega) \qquad (3.2)$$

for $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$. Note that we equip $\mathbb{R}\times\mathbb{S}^{n-1}$ with the usual one dimensional Lebesgue measure on $\mathbb{R}$ and the usual surface measure on $\mathbb{S}^{n-1}$. Thus Φ preserves the L^2 norm. The inverse of Φ satisfies $\Phi^{-1} : L^2(\mathbb{R}\times\mathbb{S}^{n-1}) \to L^2(\mathbb{R}^n)$ and is given by

$$(\Phi^{-1}\varphi)(\mathbf{x}) = r^{-n/2}\varphi\big(\ln r, \omega\big). \qquad (3.3)$$

The dilations $U(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$U(t)\psi(\mathbf{x}) := e^{tn/2}\psi(e^t\mathbf{x}) \qquad (3.4)$$

form a group of unitary operators with generator $U(t) = e^{iAt}$, where A is given by

$$iA\psi = \frac{\partial}{\partial t}U(t)\psi = (\mathbf{x}\cdot\nabla + \frac{n}{2})\psi = \frac{1}{2}(\mathbf{x}\cdot\nabla + \nabla\cdot\mathbf{x})\psi.$$

Thus

$$A = \frac{1}{i}(\mathbf{x}\cdot\nabla + \frac{n}{2}) = -iL - i\frac{n}{2}. \qquad (3.5)$$

and so

$$L = iA - \frac{n}{2},$$

where A is the self-adjoint generator of dilations in $L^2(\mathbb{R}^n)$. In particular,

$$L^*L = (-iA - \frac{n}{2})(iA - \frac{n}{2}) = A^2 + \frac{n^2}{4}. \qquad (3.6)$$

Since

$$(\Phi\psi)(s,\omega) = (U(s)\psi)(\omega) \tag{3.7}$$

for $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$, it follows from the group property of the dilations $U(\cdot)$ that

$$(\Phi(U(t)\psi))(s,\omega) = (U(s)(U(t)\psi))(\omega) = (U(s+t)\psi)(\omega) = (\Phi\psi)(s+t,\omega).$$

In particular, in the new co-ordinates given by Φ, the dilations $U(t)$ act simply as shifts by t and should be diagonalizable with the help of a Fourier transform! We now proceed to confirm this prediction.

Define $M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times S^{n-1})$ by

$$(M\psi)(\tau,\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\tau} (\Phi\psi)(s,\omega)\, ds, \tag{3.8}$$

so that $M = \mathcal{F} \circ \Phi$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. Then

$$\begin{aligned}(MU(t)\psi)(\tau,\omega) &= \frac{1}{\sqrt{2\pi}} \int e^{-is\tau} (\Phi\psi)(s+t,\omega)\, ds \\ &= \frac{e^{it\tau}}{\sqrt{2\pi}} \int e^{-is\tau} (\Phi\psi)(s,\omega)\, ds = e^{it\tau} (M\psi)(\tau,\omega).\end{aligned} \tag{3.9}$$

The map $M = \mathcal{F} \circ \Phi$ is the Mellin transformation and has an explicit representation using the group structure of $\mathbb{R}^+$ under multiplication: it is the Fourier transform on this group.

The next step is to show that

$$(MA\psi)(\tau,\omega) = \tau (M\psi)(\tau,\omega). \tag{3.10}$$

for ψ in the domain $\mathcal{D}(A)$: it follows that $\psi \in \mathcal{D}(A)$ if and only if $(\tau,\omega) \mapsto \tau(M\psi)(\tau,\omega) \in L^2(\mathbb{R} \times \mathbb{S}^{n-1})$. To see (3.10) we note that $iAe^{itA} = \partial_t U(t)$ and so, from (3.9)

$$\begin{aligned}(MiAe^{iAt}\psi)(\tau,\omega) &= (M\partial_t U(t)\psi)(\tau,\omega) = \partial_t (MU(t)\psi)(\tau,\omega) \\ &= \partial_t e^{it\tau} (M\psi)(\tau,\omega) = i\tau e^{it\tau} (M\psi)(\tau,\omega).\end{aligned}$$

Setting $t = 0$ yields (3.10).

We are now in a position to complete the proof of the theorem. We have $e^{-tL^*L} = e^{-tn^2/4} e^{-tA^2}$ and by (3.8)

$$(Me^{-tA^2}\psi)(\tau,\omega) = e^{-t\tau^2} (M\psi)(\tau,\omega). \tag{3.11}$$

So

$$e^{-tA^2} = M^{-1} e^{-t\tau^2} M.$$

Since $M = \mathcal{F} \circ \Phi$, we see that

$$e^{-tA^2} = \Phi^{-1} \circ \mathcal{F}^{-1} \big(e^{-t\tau^2} \mathcal{F} \circ \Phi \big). \tag{3.12}$$

Of course,

$$\begin{aligned}\mathcal{F}^{-1}\big(e^{-t\tau^2}M\psi\big)(\lambda,\omega) &= \mathcal{F}^{-1}\big(e^{-t\tau^2}\mathcal{F}\circ\Phi\big)(\lambda,\omega)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\tau}e^{-t\tau^2}e^{-is\tau}(\Phi\psi)(s,\omega)dsd\tau\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-t\tau^2+i(\lambda-s)\tau}d\tau\Big)(\Phi\psi)(s,\omega)\,ds\end{aligned}$$

The integral in big parentheses is a Gaussian integral which gives

$$\int_{\mathbb{R}} e^{-t\tau^2+i(\lambda-s)\tau}d\tau = \sqrt{\frac{\pi}{t}}e^{-\frac{(\lambda-s)^2}{4t}}.$$

Thus

$$\mathcal{F}^{-1}\big(e^{-t\tau^2}M\psi\big)(\lambda,\omega) = \frac{1}{\sqrt{4\pi t}}\int e^{-\frac{(\lambda-s)^2}{4t}}(\Phi\psi)(s,\omega)\,ds =: \varphi_t(\lambda,\omega)$$

and, with $\mathbf{x}=r\omega$,

$$\begin{aligned}(e^{-tA^2}\psi)(r\omega) &= (\Phi^{-1}\varphi_t)(r\omega)\\ &= r^{-n/2}\varphi_t(\ln r,\omega)\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}\int_{\mathbb{R}} e^{-\frac{(\ln r-s)^2}{4t}}(\Phi\psi)(s,\omega)\,ds.\end{aligned}$$

Since $(\Phi\psi)(s,\omega) = e^{sn/2}\psi(e^s\omega)$, we get from the change of variables $z=e^s$,

$$\begin{aligned}(e^{-tA^2}\psi)(r\omega) &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}\int_{\mathbb{R}} e^{-\frac{(\ln r-s)^2}{4t}}(\Phi\psi)(s,\omega)\,ds\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}\int_0^\infty e^{-\frac{(\ln r-\ln z)^2}{4t}}z^{\frac{n}{2}-1}\psi(z\omega)dz.\end{aligned}$$

So

$$\begin{aligned}(e^{-tL^*L}\psi)(r\omega) &= e^{-tn^2/4}(e^{-tA^2}\psi)(r\omega)\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}e^{-tn^2/4}\int_0^\infty e^{-\frac{(\ln r-\ln z)^2}{4t}}z^{\frac{n}{2}-1}\psi(z\omega)\,dz\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}e^{-tn^2/4}\int_0^\infty e^{-\frac{(\ln r-\ln z)^2}{4t}}z^{-\frac{n}{2}}\psi(z\omega)\,z^{n-1}dz\end{aligned}$$

which is (3.1).

Once it is realised that A is simply multiplication by τ in the sense of (3.10), it is clear that A is the momentum operator on $\mathbb{R}$, that is, $\Phi A\Phi^{-1}$ is given by

$$\Phi A\Phi^{-1} = -i\partial_s\otimes\mathbf{1}_{\mathbb{S}^{n-1}} \tag{3.13}$$

On using this and the functional calculus we get

$$\Phi L^*L\Phi^{-1} = (\Phi A\Phi^{-1})^2+\frac{n^2}{4} = -\partial_s^2\otimes\mathbf{1}_{S^{n-1}}+\frac{n^2}{4}. \tag{3.14}$$

Thus, $L^*L = -\Phi^{-1}\partial_s^2 \otimes \mathbf{1}_{S^{n-1}}\Phi + \frac{n^2}{4}$ and

$$e^{-tL^*L} = e^{-tn^2/4} e^{-t\Phi^{-1}\partial_s^2 \otimes \mathbf{1}_{S^{n-1}}\Phi} = e^{-tn^2/4}\Phi^{-1} e^{-t\partial_s^2 \otimes \mathbf{1}_{S^{n-1}}}\Phi \tag{3.15}$$

which is a convenient way of expressing (3.1). □

On substituting (3.2) and (3.3) and making an obvious change of variables, we obtain from (3.1) the following representation for e^{-tA^2}.

Corollary 1. *Let P_t denote e^{-tA^2}. Then*

$$\Phi P_t \Phi^{-1}\varphi(r,\omega) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \exp\{-\frac{1}{4t}(r-s)^2\}\varphi(s\omega)ds. \tag{3.16}$$

4. The main inequality

We shall denote the integral mean of a function f on $\mathbb{S}^{n-1}$, by $\mathcal{M}(f)(r)$ and when there is no danger of ambiguity, use the corresponding capital letter; thus

$$F(r) \equiv \mathcal{M}(f)(r) := |\mathbb{S}^{n-1}|^{-1}\int_{\mathbb{S}^{n-1}} f(r\omega)d\omega.$$

We have from (3.12)

$$\begin{aligned} e^{-tL^*L} &= e^{-tn^2/4}e^{-tA^2} \\ e^{-tA^2} &= \Phi^{-1}\circ\mathcal{F}^{-1}(e^{-t\tau^2}\mathcal{F}\circ\Phi). \end{aligned} \tag{4.1}$$

Therefore,

$$\Phi[e^{-tA^2}f](r,\omega) = \mathcal{F}^{-1}(e^{-t\tau^2}\mathcal{F}\circ\Phi)(F) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ir\tau - t\tau^2}(\hat{\Phi F})(\tau,\omega)d\tau \tag{4.2}$$

in which $\hat{g} := \mathcal{F}(g)$. However, the representation we use in our analysis is that given by (3.16), with now $\Phi f = g$,

$$\Phi P_t \Phi^{-1} g(r,\omega) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \exp\{-\frac{1}{4t}(r-s)^2\}g(s\omega)ds,$$

where $P_t := e^{-tA^2}$.

Define B^α to be the space of all tempered distributions g on $\mathbb{R}\times\mathbb{S}^{n-1}$ for which the norm

$$\|g\|_{B^\alpha} := \sup_{t>0}\{t^{-\alpha/2}\|\Phi e^{-tA^2}\Phi^{-1}|G|\|_{L^\infty(\mathbb{R})}\} < \infty. \tag{4.3}$$

Theorem 3. *Let $1 \le p < q < \infty$ and suppose that g is such that $\Phi A \Phi^{-1} g \equiv -i(\partial/\partial r)g \in L^p(\mathbb{R}\times\mathbb{S}^{n-1})$ and $g \in B^{\theta/(\theta-1)}, \theta = p/q$. Then there exists a positive constant C, depending on p and q such that*

$$\|G\|_{L^q(\mathbb{R})} \le C\|(\partial/\partial r)g\|^{\theta}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{1-\theta}_{B^{\theta/(\theta-1)}}. \tag{4.4}$$

Remark 2. *An intermediate result in the proof is*

$$\|G\|_{L^{q,\infty}(\mathbb{R})} \le 2^{\theta+1}\pi^{-\theta/2}|\mathbb{S}^{n-1}|^{-1}\|(\partial/\partial r)g\|^{\theta}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{1-\theta}_{B^{\theta/(\theta-1)}},$$

where $L^{q,\infty}(\mathbb{R})$ *is the weak-*L^q *space with norm*

$$\|G\|_{L^{q,\infty}(\mathbb{R})} := \left\{\sup_{u>0}[u^q\lambda(|G|\ge u)]\right\}^{1/q},$$

where $\lambda(|G|\ge u)$ *denotes the Lebesgue measure of the set* $\{r\in\mathbb{R} : |G(r)|\ge u\}$.

Remark 3. *Note that the supposition in the theorem implies that* $f = \Phi^{-1}g \in \mathcal{D}(A)$, *the domain of the operator* A *acting in* $L^2(\mathbb{R}^n)$.

To prove the theorem we first need some preliminary results on $P_t := e^{-tA^2}$.

Lemma 1. *For all* $t>0$

$$\|\Phi P_t\Phi^{-1}G]\|_{L^\infty(\mathbb{R})} \le Ct^{-1/2p}\|G\|_{L^p(\mathbb{R})}, \tag{4.5}$$

where $C \le (4\pi)^{-1/2p}(p')^{-1/2p'}$.

Proof. From (3.16) we have by Hölder's inequality that

$$\begin{aligned}\left|\Phi P_t\Phi^{-1}G(r)\right| &\le \frac{1}{\sqrt{4\pi t}}\left(\int_{\mathbb{R}} e^{-\frac{p'}{4t}(r-s)^2}ds\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}}|G(s)|^p ds\right)^{1/p}\\ &\le Ct^{-1/2p}\|G\|_{L^p(\mathbb{R})} \qquad (4.6)\end{aligned}$$

with the indicated constant. ☐

Lemma 2. *For all* $t>0$

$$\|\Phi AP_t\Phi^{-1}G\|_{L^p(\mathbb{R})} \le (\pi t)^{-1/2}\|G\|_{L^p(\mathbb{R})}$$

and similarly

$$\|\Phi AP_t\Phi^{-1}g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \le (\pi t)^{-1/2}\|g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}$$

Proof. From (3.16) we have

$$\frac{d}{dr}\left\{\Phi P_t\Phi^{-1}G(r)\right\} = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}\frac{(s-r)}{2t}\exp\left(-\frac{1}{4t}[s-r]^2\right)G(s)ds$$

and hence by Young's inequality for convolutions (see [5], Theorem V.1.2)

$$\begin{aligned}&\|\frac{d}{dr}\left\{\Phi P_t\Phi^{-1}G(r)\right\}\|_{L^p(\mathbb{R})}\\ &\le \left\{\frac{1}{\sqrt{16\pi t^3}}\int_{\mathbb{R}}|z|\exp\left(-\frac{1}{4t}z^2\right)dz\right\}\|G\|_{L^p(\mathbb{R})}\\ &= (\pi t)^{-1/2}\|G\|_{L^p(\mathbb{R})}.\end{aligned}$$

The lemma follows since

$$\begin{aligned} \frac{d}{dr}(\Phi H) &= \Phi[\frac{n}{2}H + LH] \\ &= i\Phi A H. \end{aligned} \tag{4.7}$$

□

We are now ready to prove our Theorem 3. Note that the assertion $\Phi A \Phi^{-1} g \equiv -i(\partial/\partial r) g$ follows from (4.7)(see also (3.13)). Our proof is inspired by that of Theorem 1 in [7].

Proof. **Step 1**

By homogeneity we may assume that $\|g\|_{B^{\theta/(\theta-1)}} \leq 1$, so that for all $r \in \mathbb{R}$ and $t > 0$

$$\Phi e^{-tA^2} \Phi^{-1} |G(r)| \leq t^{\theta/2(\theta-1)}.$$

For all $u > 0$ define $t_u := u^{2(\theta-1)/\theta}$ so that

$$\Phi e^{-t_u A^2} \Phi^{-1} |G(r)| \leq u. \tag{4.8}$$

Let λ denote Lebesgue measure on $\mathbb{R}$. With $P_t := e^{-tA^2}$,

$$\begin{aligned} u^q \lambda(|G| \geq 2u) &\leq u^q \lambda(|G(r) - \Phi P_{t_u} \Phi^{-1} G(r)| \geq u) \\ &\leq u^{q-p} \int_{\mathbb{R}} |G(r) - \Phi P_{t_u} \Phi^{-1} G(r)|^p dr \\ &= u^{q-p} \int_{\mathbb{R}} \left| \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} [g(r\omega) - \Phi P_{t_u} \Phi^{-1} g(r\omega)] d\omega \right|^p dr \\ &\leq u^{q-p} \frac{1}{|\mathbb{S}^{n-1}|} \|g - \Phi P_{t_u} \Phi^{-1} g\|^p_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}. \end{aligned} \tag{4.9}$$

Since $f := \Phi^{-1} g$ is assumed to lie in $\mathcal{D}(A)$, the domain of A, we have

$$\frac{\partial}{\partial t} P_t f = A^2 P_t f, \qquad P_0 f = f,$$

and consequently

$$(P_t f - f)(t) = \int_0^t A^2 P_s f ds.$$

Set $k := \Phi^{-1}h$ where $h \in C_0^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$. Then $k \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and hence lies in $\mathcal{D}(A)$. We therefore have with $\mathbf{x} = (r\omega)$

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{S}^{n-1}} h(r\omega)(\Phi P_t\Phi^{-1}g - g)(r\omega)drd\omega = \int_{\mathbb{R}^n} k(\mathbf{x})\left(P_t f(\mathbf{x}) - f(\mathbf{x})\right)d\mathbf{x}\\
&= \int_0^t \int_{\mathbb{R}^n} k(\mathbf{x})A^2P_s f(\mathbf{x})d\mathbf{x}ds\\
&= \int_0^t \int_{\mathbb{R}^n} [AP_sk](\mathbf{x})[Af](\mathbf{x})d\mathbf{x}ds\\
&= \int_0^t \int_{\mathbb{R}\times\mathbb{S}^{n-1}} [\Phi AP_s\Phi^{-1}h](r\omega)[\Phi A\Phi^{-1}g](r\omega)drd\omega ds\\
&\le \|\Phi A\Phi^{-1}g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\int_0^t \|\Phi AP_s\Phi^{-1}h\|_{L^{p'}(\mathbb{R}\times\mathbb{S}^{n-1})}ds\\
&\le 2\pi^{-\frac{1}{2}}t^{\frac{1}{2}}\|\Phi A\Phi^{-1}g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|h\|_{L^{p'}(\mathbb{R}\times\mathbb{S}^{n-1})}
\end{aligned}$$

by Lemma 2. Since $C_0^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$ is dense in $L^{p'}(\mathbb{R}\times\mathbb{S}^{n-1})$ we obtain the pseudo-Poincaré inequality (see [8])

$$\|\Phi P_t\Phi^{-1}g - g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \le 2\pi^{-\frac{1}{2}}t^{\frac{1}{2}}\|\Phi A\Phi^{-1}g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.10}$$

Thus, in (4.9),

$$\begin{aligned}
u^q\lambda(|G| \ge 2u) &\le 2^p\pi^{-\frac{p}{2}}u^{q-p}t_u^{p/2}|\mathbb{S}^{n-1}|^{-1}\|\Phi A\Phi^{-1}g\|^p_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\\
&= 2^p\pi^{-\frac{p}{2}}|\mathbb{S}^{n-1}|^{-1}\|\Phi A\Phi^{-1}g\|^p_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}, \qquad (4.11)
\end{aligned}$$

whence

$$\|G\|_{L^{q,\infty}(\mathbb{R})} \le 2^{\theta+1}\pi^{-\theta/2}|\mathbb{S}^{n-1}|^{-1}\|\Phi A\Phi^{-1}g\|^\theta_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}, \tag{4.12}$$

where $L^{q,\infty}$ denotes the weak L^q norm.

Step 2

In this step we show that the $L^{q,\infty}$ norm in (4.12) can be replaced by the L^q norm if we assume that $G \in L^q(\mathbb{R})$. We may, and shall hereafter in the proof, assume that our functions G are real-valued. Following Ledoux in [7], we write

$$5^{-q}\|G\|^q_{L^q(\mathbb{R})} = \int_0^\infty \lambda(|G| \ge 5u\})du^q \tag{4.13}$$

and for $u > 0$ define G_u by

$$G_u = (G-u)^+ \wedge ((c-1)u) + (G+u)^- \vee (-(c-1)u) \tag{4.14}$$

where $c \ge 5$, and $\wedge, \vee$ denote the minimum and maximum respectively. It follows that for $u \le |G| \le cu$

$$\frac{d}{dr}G_u = \frac{d}{dr}G \tag{4.15}$$

and is zero otherwise. Also,

$$|G| \ge 5u \Longrightarrow |G_u| \ge 4u \tag{4.16}$$

and hence

$$\int_0^\infty \lambda(|G| \geq 5u)du^q \leq \int_0^\infty \lambda(|G_u| \geq 4u)du^q. \tag{4.17}$$

We continue to assume that $\|g\|_{B^{\theta/(\theta-1)}} \leq 1$ and have $t_u = u^{2(\theta-1)/\theta}, \theta = p/q$. We have

$$\begin{aligned} |G_u| &\leq |G_u - \Phi P_{t_u}\Phi^{-1}G_u| + |\Phi P_{t_u}\Phi^{-1}[G_u - G]| + |\Phi P_{t_u}\Phi^{-1}G| \\ &\leq |G_u - \Phi P_{t_u}\Phi^{-1}G_u| + \Phi P_{t_u}\Phi^{-1}|G_u - G| + u \end{aligned} \tag{4.18}$$

since $|\Phi P_{t_u}\Phi^{-1}G| \leq \Phi P_{t_u}\Phi^{-1}|G| \leq u$. Thus $|G_u| \geq 4u$ implies that

$$|G_u - \Phi P_{t_u}\Phi^{-1}G_u| + \Phi P_{t_u}\Phi^{-1}|G_u - G| \geq 3u. \tag{4.19}$$

This in turn implies that the set $\{r : |G_u| \geq 4u\}$ is contained in $\{r : |G_u - \Phi P_{t_u}\Phi^{-1}G_u| \geq u\} \bigcup \{r : \Phi P_{t_u}\Phi^{-1}|G_u - G| \geq 2u\}$. It follows that

$$\begin{aligned} \int_0^\infty \lambda(|G_u| \geq 4u)du^q &\leq \int_0^\infty \lambda(|G_u - \Phi P_{t_u}\Phi^{-1}G_u| \geq u)du^q \\ &+ \int_0^\infty \lambda(\Phi P_{t_u}\Phi^{-1}|G_u - G| \geq 2u)du^q. \end{aligned} \tag{4.20}$$

From the pseudo-Poincaré inequality (4.10) we have, with $C = 2\pi^{-1/2}$,

$$\|G_u - \Phi P_{t_u}\Phi^{-1}G_u\|_{L^p(\mathbb{R})} \leq Ct_u^{1/2}\|\Phi A\Phi^{-1}G_u\|_{L^p(\mathbb{R})} \tag{4.21}$$

and hence, on using (4.7), (4.15) and (4.21), and recalling that $t_u = u^{2(\theta-1)/\theta}$, so that $u^{-p}t_u^{p/2} = u^{-q}$,

$$\begin{aligned} \lambda(|G_u - \Phi P_{t_u}\Phi^{-1}G_u| \geq u) &\leq u^{-p}\int_0^\infty |G_u - \Phi P_{t_u}\Phi^{-1}G_u|^p dr \\ &\leq Cu^{-p}t_u^{p/2}\|\Phi A\Phi^{-1}G_u\|^p_{L^p(\mathbb{R})} \\ &= Cu^{-q}\|\frac{d}{dr}G_u\|^p_{L^p(\mathbb{R})} \\ &= Cu^{-q}\int_{u<|G|<cu}|\frac{d}{dr}G|^p dr \\ &= Cu^{-q}\int_{u<|G|<cu}|\Phi A\Phi^{-1}G|^p dr. \end{aligned} \tag{4.22}$$

Hence

$$\begin{aligned}
&\int_0^\infty \lambda(|G_u - \Phi P_{t_u}\Phi^{-1}G_u| \geq u)du^q \\
&\leq C\int_0^\infty \left\{u^{-q}\int_{u<|G|<cu} |\Phi A\Phi^{-1}G|^p dr\right\} du^q \\
&= C\int_{\mathbb{R}} |\Phi A\Phi^{-1}G(r)|^p \left\{\int_{|G|/c}^{|G|} u^{-q}du^q\right\} dr \\
&= Cq\ln c\|\Phi A\Phi^{-1}G\|^p_{L^p(\mathbb{R})} \\
&\leq Cq\ln c\frac{1}{|\mathbb{S}^{n-1}|}\|(\partial/\partial r)g\|^p_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \qquad (4.23)
\end{aligned}$$

by (4.7).

Next we consider $\lambda(\Phi P_{t_u}\Phi^{-1}|G_u - G| \geq 2u)$. First, we claim that

$$\Phi P_{t_u}\Phi^{-1}|G_u - G| \leq u + \Phi P_{t_u}\Phi^{-1}|G|\chi_{\{|G|\geq cu\}}, \qquad (4.24)$$

where χ_I denotes the characteristic function of the set I. We have from (4.14)

$$\begin{aligned}
|G_u - G| &\leq |G_u - G|\chi_{\{|G|\leq cu\}} + |G_u - G|\chi_{\{|G|\geq cu\}} \\
&\leq u + |G_u - G|\chi_{\{|G|\geq cu\}}. \qquad (4.25)
\end{aligned}$$

Hence, from (3.16),

$$\begin{aligned}
&\Phi P_{t_u}\Phi^{-1}|G_u - G| \leq \frac{u}{\sqrt{4\pi t_u}}\int_{\mathbb{R}} \exp\{-\frac{1}{4t_u}(r-s)^2\}ds \\
&+ \frac{1}{\sqrt{4\pi t_u}}\int_{\mathbb{R}} \exp\{-\frac{1}{4t_u}(r-s)^2\}|G - G_u|\chi_{\{|G|\geq cu\}}ds \\
&= u + \frac{1}{\sqrt{4\pi t_u}}\int_{\mathbb{R}} \exp\{-\frac{1}{4t_u}(r-s)^2\}|G - G_u|\chi_{\{|G|\geq cu\}}ds. \\
&\qquad (4.26)
\end{aligned}$$

For $|G| \geq cu$, we have from the construction of G_u in (4.14) that

$$|G - G_u| \leq |G| \qquad (4.27)$$

and hence on substituting in (4.26) we get

$$\begin{aligned}
\Phi P_{t_u}\Phi^{-1}|G_u - G| &\leq u + \frac{1}{\sqrt{4\pi t_u}}\int_{\mathbb{R}} \exp\{-\frac{1}{4t_u}(r-s)^2\}|G|\chi_{\{|G|\geq cu\}}ds \\
&= u + \Phi P_{t_u}\Phi^{-1}|G|\chi_{\{|G|\geq cu\}}, \qquad (4.28)
\end{aligned}$$

as claimed in (4.24). This gives

$$\begin{aligned}
&\int_0^\infty \lambda(\Phi P_{t_u}\Phi^{-1}|G_u - G| \geq 2u)du^q \\
&\leq \int_0^\infty \lambda(\Phi P_{t_u}\Phi^{-1}|G|\chi_{\{|G|\geq cu\}} \geq u)du^q \\
&\leq \int_0^\infty u^{-1}\left(\int_{\mathbb{R}} \Phi P_{t_u}\Phi^{-1}|G|\chi_{\{|G|\geq cu\}}dr\right)du^q \\
&= \int_0^\infty \frac{1}{\sqrt{4\pi t_u}}\int_{\mathbb{R}}\left[\int_0^\infty \exp\{-\frac{1}{4t_u}(r-s)^2\}|G|\chi_{\{|G|\geq cu\}}ds\right]dr\frac{du^q}{u} \\
&\leq \int_0^\infty u^{-1}\int_0^\infty |G|\chi_{\{|G|\geq cu\}}ds du^q \\
&= q\int_0^\infty |G|\left(\int_0^{|G|/c} u^{q-2}du\right)ds \\
&= \frac{q}{(q-1)c^{q-1}}\|G\|^q_{L^q(\mathbb{R})}. \qquad (4.29)
\end{aligned}$$

We have therefore shown that

$$5^{-q}\|G\|^q_{L^q(\mathbb{R})} \leq Cq\ln c\|(\partial/\partial r)g\|^p_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} + \frac{q}{(q-1)c^{q-1}}\|G\|^q_{L^q(\mathbb{R})}$$

which on choosing c large enough yields (4.4) under the additional assumption $G \in L^q(\mathbb{R})$.

Step 3

The final step is to remove the assumption $G \in L^q(\mathbb{R})$ in Step 2. We again follow Ledoux's approach and define

$$N_\varepsilon(G) = \int_\varepsilon^{1/\varepsilon} \lambda(|G| \geq 5u)d(u^q) < \infty.$$

From (4.17), (4.20), (4.23) and (4.29) it is seen that

$$N_\varepsilon(G) \leq Cq\ln c\|(\partial/\partial r)g\|^p_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} + \int_\varepsilon^{1/\varepsilon}\frac{1}{u}\left(\int |G|\chi_{\{|G|>cu\}}d\lambda\right)d(u^q). \qquad (4.30)$$

We shall use the fact that

$$\int |G|\chi_{\{|G|>cu\}}d\lambda = -\int_{cu}^\infty \alpha d\lambda(\alpha) \qquad (4.31)$$

where

$$\lambda(\alpha) := \lambda\{x : |G(x)| > \alpha\}.$$

On integration by parts, we have for all $\Lambda > cu$, that

$$\begin{aligned}
-\int_{cu}^\Lambda \alpha d\lambda(\alpha) &= -[\alpha\lambda(\alpha)]^\Lambda_{cu} + \int_{cu}^\infty \lambda(\alpha)d\alpha \\
&\leq cu\lambda(cu) + \int_{cu}^\infty \lambda(\alpha)d\alpha
\end{aligned}$$

and hence

$$\int |G|\chi_{\{|G|>cu\}} d\lambda \leq cu\lambda(cu) + \int_{cu}^{\infty} \lambda(\alpha)d\alpha \tag{4.32}$$

From this we infer that

$$\begin{aligned} I &:= \int_{\varepsilon}^{1/\varepsilon} \frac{1}{u}\left(\int |G|\chi_{\{|f|>cu\}} d\lambda\right) d(u^q) \\ &\leq c\int_{\varepsilon}^{1/\varepsilon} \lambda(cu)d(u^q) + \int_{\varepsilon}^{1/\varepsilon}\left(\int_{cu}^{\infty} \lambda(\alpha)d\alpha\right) qu^{q-2}du \\ &= c\int_{\varepsilon}^{1/\varepsilon} \lambda(cu)d(u^q) + I_1 \end{aligned} \tag{4.33}$$

say. We now apply Fubini's Theorem to I_1.

$$\begin{aligned} I_1 &= \int_{\alpha=c\varepsilon}^{c/\varepsilon} \lambda(\alpha)d\alpha \int_{u=\varepsilon}^{\alpha/c} qu^{q-2}du \\ &+ \int_{\alpha=c/\varepsilon}^{\infty} \lambda(\alpha)d\alpha \int_{u=\varepsilon}^{1/\varepsilon} qu^{q-2}du \\ &= c\int_{t=\varepsilon}^{1/\varepsilon} \lambda(ct)dt \left[\frac{q}{(q-1)}u^{q-1}\right]_{\varepsilon}^{t} + c\int_{t=1/\varepsilon}^{\infty} \lambda(ct)dt \left[\frac{q}{(q-1)}u^{q-1}\right]_{\varepsilon}^{1/\varepsilon} \\ &\leq \frac{cq}{(q-1)}\int_{\varepsilon}^{1/\varepsilon} t^{q-1}\lambda(ct)dt + \frac{cq}{(q-1)}\frac{1}{\varepsilon^{q-1}}\int_{1/\varepsilon}^{\infty} \lambda(ct)dt \\ &= \frac{c}{(q-1)}\int_{\varepsilon}^{1/\varepsilon} \lambda(ct)d(t^q) + \frac{cq}{(q-1)}\frac{1}{\varepsilon^{q-1}}\int_{1/\varepsilon}^{\infty} \lambda(ct)dt. \end{aligned} \tag{4.34}$$

It follows from (4.33) and (4.34) that

$$I \leq \frac{cq}{(q-1)}\int_{\varepsilon}^{1/\varepsilon} \lambda(ct)d(t^q) + \frac{cq}{(q-1)}\frac{1}{\varepsilon^{q-1}}\int_{1/\varepsilon}^{\infty} \lambda(ct)dt. \tag{4.35}$$

On setting $t = (c/5)u, \varepsilon = (5/c)\tilde{\varepsilon}$ we have

$$\begin{aligned} \frac{cq}{(q-1)}\int_{\varepsilon}^{1/\varepsilon} \lambda(ct)d(t^q) &= \frac{q}{(q-1)}\frac{5^q}{c^{q-1}}N_{\tilde{\varepsilon}}(G) \\ &\leq \frac{q}{(q-1)}\frac{5^q}{c^{q-1}}N_{\varepsilon}(G) \end{aligned} \tag{4.36}$$

since $\tilde{\varepsilon} \geq \varepsilon$. We also have in (4.35)

$$\begin{aligned} \int_{1/\varepsilon}^{\infty} \lambda(|G| > cu)du &= \int_{1/\varepsilon}^{\infty} (cu)^q\lambda(|G| > cu)(cu)^{-q}du \\ &\leq \frac{1}{c^q}\|G\|_{q,\infty}^q \int_{1/\varepsilon}^{\infty} u^{-q}du \\ &= \frac{\varepsilon^{q-1}}{c^q(q-1)}\|G\|_{L^{q,\infty}(\mathbb{R})}^q \end{aligned}$$

and so

$$\frac{cq}{(q-1)\varepsilon^{q-1}}\int_{1/\varepsilon}^{\infty}\lambda(|G|>cu)du \le \frac{q}{(q-1)^2c^{q-1}}\|G\|^q_{L^{q,\infty}(\mathbb{R})}. \tag{4.37}$$

We therefore have from (4.30)

$$\begin{aligned} N_\varepsilon(G) &\le Cq\ln c\|(\partial/\partial r)g\|^p_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} + \frac{q}{(q-1)}\frac{5^q}{c^{q-1}}N_\varepsilon(G) \\ &+ \frac{q}{(q-1)^2c^{q-1}}\|G\|^q_{L^{q,\infty}(\mathbb{R})}. \end{aligned} \tag{4.38}$$

On choosing c large enough it follows that $\sup_{\varepsilon>0}N_\varepsilon(G)<\infty$ and so $G\in L^q(\mathbb{R})$. The proof is therefore complete. □

The theorem has two natural corollaries featuring the Hardy-type inequality (2.1), the first an inequality of Sobolev type , and the second of Gagliardo-Nirenberg type.

Corollary 2. *Let* $p^* := np/(n-p), 1\le p<n$, *and suppose* $g, (\partial/\partial r)g \in L^p(\mathbb{R}\times\mathbb{S}^{n-1})$. *Then*

$$\|G\|_{L^{p^*}(\mathbb{R})} \le C\|(\partial/\partial r)g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{(n-1)/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.39}$$

If G *is supported in* $[-\Lambda,\Lambda]$, *then*

$$\|G\|_{L^{p^*}(\mathbb{R})} \le C\Lambda^{(n-1)/n}\|(\partial/\partial r)g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.40}$$

Proof. From Lemma 1

$$\begin{aligned} t^{-\theta/2(\theta-1)}\|\Phi P_t\Phi^{-1}|G|\|_{L^\infty(\mathbb{R})} &\le Ct^{-\theta/2(\theta-1)-1/2p}\|G\|_{L^p(\mathbb{R})} \\ &\le C\|G\|_{L^p(\mathbb{R})} \\ &\le C\|g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1}})\end{aligned}$$

if $\theta=p/q, q=p(p+1)$. Hence from Theorem 3

$$\|G\|_{L^{p(p+1)}(\mathbb{R})} \le C\|(\partial/\partial r)g\|^{1/(p+1)}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{p/(p+1)}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.41}$$

Thus $G\in L^{p(p+1)}(\mathbb{R})\cap L^p(\mathbb{R})$, and since

$$\frac{np}{(n-p)} = \frac{p(p+1)}{(n-p)} + \frac{p(n-p-1)}{(n-p)}$$

we have by Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}}|G|^{p^*}dr &\le \left(\int_{\mathbb{R}}|G|^{p(p+1)}dr\right)^{1/(n-p)}\left(\int_{\mathbb{R}}|G|^p dr\right)^{(n-p-1)/(n-p)} \\ &\le \left(\int_{\mathbb{R}}|G|^{p(p+1)}dr\right)^{1/(n-p)}\left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{R}\times\mathbb{S}^{n-1}}|g|^p dr d\omega\right)^{(n-p-1)/(n-p)}. \end{aligned}$$

Hence, from (4.41),

$$\begin{aligned} \|G\|_{L^{p^*}(\mathbb{R})} &\le C\|G\|^{(p+1)/n}_{L^{p(p+1)}(\mathbb{R})}\|g\|^{(n-p-1)/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \\ &\le C\|(\partial/\partial r)g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{(n-1)/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \end{aligned}$$

The inequality (4.40) follows on using Hölder's inequality to give

$$\|G\|_{L^p(\mathbb{R})} \leq \|G\|_{L^{p^*}(\mathbb{R})}(2\Lambda)^{(1/p)-(1/p^*)}$$

and then substituting in

$$\|G\|_{L^{p(p+1)}(\mathbb{R})} \leq C\|(\partial/\partial r)g\|^{1/(p+1)}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|G\|^{p/(p+1)}_{L^p(\mathbb{R})}$$

which is proved in the course of establishing (4.41). □

Corollary 3. *Let* $1 \leq p < q < \infty, m = (q/p) - 1$, *and suppose that* $(\partial/\partial r)g \in L^p(\mathbb{R}\times\mathbb{S}^{n-1}), g \in L^m(\mathbb{R}\times\mathbb{S}^{n-1})$. *Then*

$$\|G\|_{L^q(\mathbb{R})} \leq C\|(\partial/\partial r)g\|^{p/q}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{1-p/q}_{L^m(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.42}$$

Proof. From Lemma 1, with $\theta = p/q$ and $m = q/p - 1$,

$$\begin{aligned} t^{-\theta/2(\theta-1)}\|\Phi P_t\Phi^{-1}|G|\|_{L^\infty(\mathbb{R})} &\leq Ct^{-\theta/2(\theta-1)-1/2m}\|G\|_{L^m(\mathbb{R})} \\ &\leq C\|g\|_{L^m(\mathbb{R}\times\mathbb{S}^{n-1})} \end{aligned}$$

and this yields (4.42). □

The cases $p = 2$ of Corollaries 2 and 3 are of special interest.

Corollary 4. *Let* f *be such that* $f, Lf \in L^2(\mathbb{R}^n)$, *where* $L = \mathbf{x}\cdot\nabla$. *Then for* $n > 2$,

$$\begin{aligned} \|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} &\leq C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \|f\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{4.43}$$

where $F = \mathcal{M}(f), 2^* = 2n/(n-2)$ *and* $d\mu = r^{n-1}dr$.

Proof. On using the facts that $\Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ is an isometry and, with $g := \Phi f$,

$$\begin{aligned} \|(\partial/\partial r)g\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{n-1})} &= \|\Phi A\Phi^{-1}g\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{n-1})} \\ &= \|Af\|^2_{L^2(\mathbb{R}^n)} \\ &= \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)} \end{aligned}$$

since $A^2 = L^*L - (n^2/4)$ from (3.6), it follows from (4.39) that

$$\begin{aligned} \|\mathcal{M}(\Phi f)\|^2_{L^{2^*}(\mathbb{R})} &\leq C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \|f\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The corollary follows since

$$\|\mathcal{M}(\Phi f)\|_{L^{2^*}(\mathbb{R})} = \|rF(r)\|_{L^{2^*}(\mathbb{R}^+;d\mu)}.$$

□

Corollary 5. *Let $h, \nabla h \in L^2(\mathbb{R}^n), n \geq 3$. Then there exists a positive constant C depending only on n such that*

$$\begin{aligned} \|\mathcal{M}(h)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} &\leq C\big\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \big(\frac{n-2}{2}\big)^2\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\big\}^{1/n} \\ &\times \big\{\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\big\}^{1-1/n}. \end{aligned} \quad (4.44)$$

Hence, for any $\varepsilon > 0$,

$$\begin{aligned} \varepsilon^{1-1/n}\|\mathcal{M}(h)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} &\leq C\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} \\ &- [(\frac{n-2}{2})^2 - \varepsilon]\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\}. \end{aligned} \quad (4.45)$$

Proof. Since $n \geq 3$, we have that $f := h/|\cdot| \in L^2(\mathbb{R}^n)$. We claim that $Lf \in L^2(\mathbb{R}^n)$. For

$$\begin{aligned} |\nabla(|\mathbf{x}|f)|^2 &= \left|\frac{\mathbf{x}}{|\mathbf{x}|}f + |\mathbf{x}|\nabla f\right|^2 \\ &= |f|^2 + (|\mathbf{x}||\nabla f|)^2 + 2\mathrm{Re}[\bar{\mathrm{f}}(\mathbf{x}\cdot\nabla)\mathrm{f}] \end{aligned}$$

and, on integration by parts, initially for $f \in C_0^\infty(\mathbb{R}^n)$ and then by the usual continuity argument,

$$\begin{aligned} \int_{\mathbb{R}^n} \overline{f}(\mathbf{x}\cdot\nabla)f d\mathbf{x} &= \sum_{j=1}^n \int_{\mathbb{R}^n} x_j\overline{f}\frac{\partial f}{\partial x_j}d\mathbf{x} \\ &= -\sum_{j=1}^n \int_{\mathbb{R}^n} f\left\{\overline{f} + x_j\frac{\partial\overline{f}}{\partial x_j}\right\}d\mathbf{x} \\ &= -\int_{\mathbb{R}^n}\left\{n|f|^2 + f(\mathbf{x}\cdot\nabla)\overline{f}\right\}d\mathbf{x}. \end{aligned}$$

This gives

$$2\mathrm{Re}\int_{\mathbb{R}^n}[\bar{\mathrm{f}}(\mathbf{x}\cdot\nabla)\mathrm{f}]\mathrm{d}\mathbf{x} = -\mathrm{n}\int_{\mathbb{R}^n}|\mathrm{f}|^2\mathrm{d}\mathbf{x}$$

and hence

$$\begin{aligned} \int_{\mathbb{R}^n}|\nabla(|\mathbf{x}|f)|^2 d\mathbf{x} &= \int_{\mathbb{R}^n}(|\mathbf{x}||\nabla f|)^2 d\mathbf{x} - (n-1)\int_{\mathbb{R}^n}|f|^2 d\mathbf{x} \\ &\geq \int_{\mathbb{R}^n}|Lf|^2 d\mathbf{x} - (n-1)\int_{\mathbb{R}^n}|f|^2 d\mathbf{x} \end{aligned} \quad (4.46)$$

which confirms our claim. On substituting (4.46) and $f = h/|\cdot|$ in Corollary 4 we get

$$\begin{aligned} \|\mathcal{M}(h)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} &\leq C\Big\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} + (n-1)\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)} \\ &- (n^2/4)\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\Big\}^{1/n}\|h/|\cdot|\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)} \end{aligned}$$

which yields (4.44). The inequality (4.45) follows from

$$n[\varepsilon/(n-1)]^{1-1/n}ab \leq a^n + \varepsilon b^{n/(n-1)}$$

which is a consequence of Young's inequality. □

The inequality (4.45) is implied by Stubbe's inequality (1.7). For on setting $\delta = (n-2)^2/4 - \varepsilon$ in (4.45) we have

$$\|\mathcal{M}(h)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C[\frac{(n-2)^2}{4} - \delta]^{-\frac{(n-1)}{n}} \{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \delta\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\}. \tag{4.47}$$

Since

$$\|\mathcal{M}(h)\|^{2^*}_{L^{2^*}(\mathbb{R}^+;d\mu)} \le \frac{1}{|\mathbb{S}^{n-1}|}\|h\|^{2^*}_{L^{2^*}(\mathbb{R}^n)}$$

by Hölder's inequality, it follows that (4.47) is a consequence of (1.7).

If in (4.40) $g = \Phi f$, where f is supported in the annulus $A(1/R, R) := \{\mathbf{x} \in \mathbb{R}^n : 1/R \le |\mathbf{x}| \le R\}$, then G is supported in the interval $[-\ln R, \ln R]$ and we have

Corollary 6. *Let f in Corollary 4 be supported in the annulus $A(1/R, R)$. Then*

$$\|r\mathcal{M}(f)(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C(\ln R)^{\frac{2(n-1)}{n}} \left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}. \tag{4.48}$$

On putting $f = h/|\cdot|$ in (4.48) we have as in the proof of Corollary 5

Corollary 7. *Let h in Corollary 5 have support in the annulus $A(1/R, R)$. Then*

$$\|\mathcal{M}(h)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C(\ln R)^{\frac{2(n-1)}{n}} \left\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \frac{(n-2)^2}{4}\|\frac{h}{|\cdot|}\|^2_{L^2(\mathbb{R}^n)}\right\}. \tag{4.49}$$

Finally we have the following $p = 2$ case of Corollary 3.

Corollary 8. *Let $2 < q < \infty$ and $m = q/2 - 1$. Then, if f is such that $f, Lf \in L^2(\mathbb{R}^n)$ and $\int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}} |f(s\omega)|^m s^{n(\frac{m}{2}-1)} ds d\omega < \infty$, we have that $\int_{\mathbb{R}^+} |f(s\omega)|^q s^{nm} ds d\omega < \infty$ and*

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}} |f(s\omega)|^q s^{nm} ds d\omega \quad &\le \quad C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^2 \\ &\times \quad \left\{\int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}} |f(s\omega)|^m s^{n(\frac{m}{2}-1)} ds d\omega\right\}^2\end{aligned} \tag{4.50}$$

Proof. Corollary 3 with $p = 2$ yields

$$\begin{aligned}\|\mathcal{M}(\Phi f)\|_{L^q(\mathbb{R})} \quad &\le \quad C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{2/q} \\ &\times \quad \|\Phi f\|^{1-2/q}_{L^m(\mathbb{R})}.\end{aligned}$$

Since

$$\|\mathcal{M}(\Phi f)\|_{L^q(\mathbb{R})} = |\mathbb{S}^{n-1}|^{-1}\|s^{nm} f\|_{L^q(\mathbb{R}\times\mathbb{S}^{n-1})}$$

and

$$\|\Phi f\|^m_{L^m(\mathbb{R}\times\mathbb{S}^{n-1})} = \int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}} |f(s)|^m s^{n(\frac{m}{2}-1)} ds d\omega$$

the corollary follows. ☐

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