NEW BOUNDS ON THE LIEB-THIRRING CONSTANTS

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ABSTRACT. Improved estimates on the constants $L_{\gamma,d}$, for $1/2 < \gamma < 3/2$, $d \in \mathbb{N}$ in the inequalities for the eigenvalue moments of Schrödinger operators are established.

1. INTRODUCTION

Let us consider a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$-\Delta + V, \tag{1.1}$$

where V is a real-valued function. The inequalities

$$\operatorname{tr}(-\Delta + V)_-^\gamma \le L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}\, dx\,, \tag{1.2}$$

are known as Lieb-Thirring bounds and hold true with finite constants $L_{\gamma,d}$ if and only if $\gamma \ge 1/2$ for $d = 1$, $\gamma > 0$ for $d = 2$ and $\gamma \ge 0$ for $d \ge 3$. Here and in the following, $A_\pm = (|A| \pm A)/2$ denote the positive and negative parts of a self-adjoint operator A. The case $\gamma > (1 - d/2)_+$ was shown by Lieb and Thirring in [21]. The critical case $\gamma = 0$, $d \ge 3$ is known as the Cwikel-Lieb-Rozenblum inequality, see [8, 19, 22] and also [18, 7]. The remaining case $\gamma = 1/2$, $d = 1$ was verified in [25].

It is known that as soon as $V \in L^{\gamma+d/2}(\mathbb{R}^d)$ and the constant $L_{\gamma,d}$ is finite, then we have Weyl's asymptotic formula

$$\begin{aligned}\lim_{\alpha\to+\infty} \frac{1}{\alpha^{\gamma+\frac{d}{2}}} \operatorname{tr}(-\Delta + \alpha V)_-^\gamma &= \lim_{\alpha\to+\infty} \frac{1}{\alpha^{\gamma+\frac{d}{2}}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|\xi|^2 + \alpha V)_-^\gamma \frac{dx\,d\xi}{(2\pi)^d} \\ &= L^{cl}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}\, dx\,, \end{aligned} \tag{1.3}$$

where the so-called classical constant $L^{cl}_{\gamma,d}$ is defined by

(1.4)
$$L^{cl}_{\gamma,d} = (2\pi)^{-d} \int_{\mathbb{R}^d} (|\xi|^2 - 1)_-^\gamma\, d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma + \frac{d}{2} + 1)}\,, \quad \gamma \ge 0\,.$$

This immediately implies $L^{cl}_{\gamma,d} \le L_{\gamma,d}$.

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Until recently the sharp values of $L_{\gamma,d}$ were known only for $\gamma \geq 3/2$, $d = 1$, (see [21, 1]), where they coincide with $L^{cl}_{\gamma,d}$. In [17] Laptev and Weidl extended this result to all dimensions. They proved that $L_{\gamma,d} = L^{cl}_{\gamma,d}$, for $\gamma \geq 3/2$, $d \in \mathbb{N}$. Recently, Hundertmark, Lieb and Thomas showed in [15] that the sharp value of $L_{1/2,1}$ is equal to $1/2$.

The purpose of this paper is to give some new bounds on the constants $L_{\gamma,d}$ for $1/2 < \gamma < 3/2$ and all $d \in \mathbb{N}$ (see §4). In particular, one of our main results given in Theorem 4.1, says that

$$L_{\gamma,d} \leq 2L^{cl}_{\gamma,d}, \qquad 1 \leq \gamma < 3/2, \quad d \in \mathbb{N}, \tag{1.5}$$

whereas for large dimensions it was only known that $L_{\gamma,d} \leq C\sqrt{d}\, L^{cl}_{\gamma,d}$ with some constant $C > 0$.

For the important case $\gamma = 1$, $d = 3$ we have $L_{1,3} \leq 2L^{cl}_{1,3} < 0.013509$ compared with $L_{1,3} < 5.96677L^{cl}_{1,3} < 0.040303$ obtained in [20] and its improvement $L_{1,3} < 5.21803L^{cl}_{1,3} < 0.035246$ obtained in [5].

Note also that our estimates on the constant $L_{\gamma,d}$ imply that $L_{1,d} \leq 2L^{cl}_{1,d} < L^{cl}_{0,d}$ as was conjectured in [23].

In order to obtain our results we give a version of the proof obtained in [15] for matrix-valued potentials (see §3). Note that E.H.Lieb has informed us that the original proof obtained in [15] also works for matrix-valued potentials. After that in §4 we apply the equality $L_{0,d} = L^{cl}_{0,d}$, for $\gamma \geq 3/2$ and $d \in \mathbb{N}$ shown in [17] by using the "lifting" argument with respect to the dimension d suggested in [16]. The same arguments as in [17] yield the corresponding inequalities for Schrödinger operators with magnetic fields.

In §5 we recover the matrix-valued version of the Buslaev-Faddeev-Zakharov trace formulae obtained in [17] and find some new two sides spectral inequalities for one-dimensional Schrödinger operators with operator-valued potentials.

Finally, we are very grateful to L.E.Thomas who was also involved in the new proof of Theorem 3.1 and has written § 3.4 as well as reading the text of the paper and making many valuable remarks.

2. Notation and Auxiliary Material

Let $\mathbf{G}$ be a separable Hilbert space with the norm $\|\cdot\|_{\mathbf{G}}$ and the scalar product $\langle\cdot,\cdot\rangle_{\mathbf{G}}$ and let $0_{\mathbf{G}}$ and $1_{\mathbf{G}}$ be the zero and the identity operator on $\mathbf{G}$. Denote by $\mathcal{B}(\mathbf{G})$ the Banach space of all bounded operators on $\mathbf{G}$ and by $\mathcal{K}(\mathbf{G})$ the (separable) ideal of all compact operators. Let $\mathcal{S}_1(\mathbf{G})$ and $\mathcal{S}_2(\mathbf{G})$ be the classes of trace and Hilbert-Schmidt operators on $\mathbf{G}$ respectively. For a nonnegative operator $A \in \mathcal{K}(\mathbf{G})$

$$\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq 0$$

is the ordered sequence of its eigenvalues (including multiplicities). We use the symbol "tr" to denote traces of operators (matrices) in different Hilbert spaces.

The Hilbert space $\mathbf{H} = L^2(\mathbb{R}^d, \mathbf{G})$ is the space of all measurable functions $u : \mathbb{R}^d \to \mathbf{G}$ such that

$$\|u\|^2_{\mathbf{H}} := \int_{\mathbb{R}^d} \|u\|^2_{\mathbf{G}}\, dx \; < \infty.$$

The Sobolev space $H^1(\mathbb{R}^d, \mathbf{G})$ consists of all functions $u \in \mathbf{H}$ whose norm

$$\|u\|^2_{H^1(\mathbb{R}^d,\mathbf{G})} = \sum_{k=1}^{d} \|\partial u/\partial x_k\|^2_{\mathbf{H}} + \|u\|^2_{\mathbf{H}}$$

is finite. Obviously the quadratic form

$$h[u,u] = \sum_{k=1}^{d} \|\partial u/\partial x_k\|^2_{\mathbf{H}}$$

is closed in $L^2(\mathbb{R}^d, \mathbf{G})$ on the domain $u \in H^1(\mathbb{R}^d, \mathbf{G})$. Let

$$V(\cdot) : \mathbb{R}^d \to B(\mathbf{G})$$

be an operator-valued function satisfying

$$\|V(\cdot)\|_{B(\mathbf{G})} \in L^p(\mathbb{R}^d) \tag{2.1}$$

for some finite p with

$$\begin{aligned} p \geq&\ 1 \quad \text{if} \quad d = 1\,, \\ p >&\ 1 \quad \text{if} \quad d = 2\,, \\ p \geq&\ d/2 \quad \text{if} \quad d \geq 3\,. \end{aligned}$$

Then the quadratic form

$$v[u,u] = \int_{\mathbb{R}^d} \langle Vu, u\rangle_{\mathbf{G}}\; dx$$

is bounded with respect to $h[\cdot,\cdot]$ and thus the form

$$h[u,u] + v[u,u] \tag{2.2}$$

is closed and semi-bounded from below on $H^1(\mathbb{R}^d, \mathbf{G})$. It generates the self-adjoint operator

$$Q = -\,(\Delta \otimes \mathbf{1}_{\mathbf{G}}) + V(x) \tag{2.3}$$

in $L^2(\mathbb{R}^d, \mathbf{G})$. It is not difficult to see, that if the operator $V(x)$ belongs to $\mathcal{K}(\mathbf{G})$ for a.e. $x \in \mathbb{R}^d$ and satisfies the condition (2.1), then the negative spectrum

$$-E_1 \leq -E_2 \leq \cdots \leq -E_n \leq \cdots < 0$$

of the operator Q is discrete.

3. An upper bound for the eigenvalue moment in the critical case $d = 1$ and $\gamma = 1/2$.

3.1. **A sharp Lieb-Thirring inequality for $d = 1$ and $\gamma = 1/2$.**

In this section we give a version of the proof from [15] which will be applied to the Schrödinger operators with operator-valued potentials. The main result of this section is the following statement:

Theorem 3.1. *Let $V(x)$ be a nonpositive operator-valued function, such that $V(x) \in \mathcal{S}_1(\mathbf{G})$ for a.e. $x \in \mathbb{R}$ and $\operatorname{tr} V_-(\cdot) \in L^1(\mathbb{R})$. Then*

$$\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V\Big)_-^{1/2} = \sum_j \sqrt{E_j} \le \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{tr} V_-\, dx\,. \tag{3.1}$$

Remark. The constant $L_{1/2,1} = 1/2 = 2L^{\mathrm{cl}}_{1/2,1}$ is the best possible. Indeed, $1/2$ is achieved by the operator of rank one $V(x) = \delta(x)\,\langle\cdot, e\rangle\, e$, where $e \in \mathbf{G}$ and δ is Dirac's δ-function (see [15]).

We follow the strategy of [15] quite closely but give a different proof of the monotonicity lemma.

3.2. **Monotonicity Lemma.**

In order to prove the monotonicity lemma we need an auxiliary "majorization" result. Let $A \in \mathcal{K}(\mathbf{G})$ and let us denote

$$\|A\|_n = \sum_{j=1}^{n} \sqrt{\lambda_j(A^*A)}.$$

Then by Ky-Fan's inequality (see for example [12, Lemma 4.2]) the functionals $\|\cdot\|_n$, $n = 1, 2, \ldots$, are norms on $\mathcal{K}(\mathbf{G})$ and thus for any unitary operator U in $\mathbf{G}$ we have

$$\|U^*AU\|_n = \|A\|_n.$$

Definition 3.2. *Let A, B be two compact operators on $\mathbf{G}$. We say that A majorizes B or $B \prec A$, iff*

$$\|B\|_n \le \|A\|_n \qquad \textit{for all } n \in \mathbb{N}\,.$$

Lemma 3.3 (Majorization)**.** *Let A be a nonnegative compact operator $\mathbf{G}$, $\{U(\omega)\}_{\omega\in\Omega}$ be a family of unitary operators on $\mathbf{G}$, and let g be a probability measure on Ω. Then the operator*

$$B := \int_\Omega U^*(\omega) A U(\omega)\, g(d\omega)$$

is majorized by A.

Proof. This is a simple consequence of the triangle inequality

$$\|B\|_n \le \int_\Omega \|U^*(\omega)AU(\omega)\|_n\, g(d\omega) = g(\Omega)\|A\|_n = \|A\|_n\,.$$

■

Remark. The notion of majorization is well-known in matrix theory (see [3]). For finite dimensional Hilbert spaces **G** even the converse statement of Lemma 3.3 is true, cf. [2, Theorem 7.1]:
If A and B are nonnegative matrices and $\operatorname{tr} A = \operatorname{tr} B$, then the condition $B \prec A$ implies that there exist unitary matrices U_j and $t_j > 0$, $j = 1, \dots, N$, such that

$$\sum_{j=1}^N t_j = 1, \qquad B = \sum_{j=1}^N t_j U_j^* A U_j.$$

Let $W(\cdot) : \mathbb{R} \to \mathcal{S}_2(\mathbf{G})$ be an operator-valued function and let $\|W(\cdot)\|_{\mathcal{S}_2} \in L^2(\mathbb{R})$. Denote

$$\mathcal{L}_\varepsilon := W^* \left[2\varepsilon \Big(-\frac{d^2}{dx^2} + \varepsilon^2 \Big)^{-1} \otimes \mathbf{1}_{\mathbf{G}} \right] W\,. \tag{3.2}$$

Obviously, $\mathcal{L}_\varepsilon$ is a nonnegative, trace class operator on $L^2(\mathbb{R}, \mathbf{G})$, its trace is independent of ε, $0 \le \varepsilon < \infty$ and equals $\operatorname{tr} \mathcal{L}_\varepsilon = \int \|W(x)\|^2_{\mathcal{S}_2}\, dx$.

Lemma 3.4 (Monotonicity)**.** *The operator $\mathcal{L}_\varepsilon$ is majorized by $\mathcal{L}_{\varepsilon'}$*

$$\mathcal{L}_\varepsilon \prec \mathcal{L}_{\varepsilon'}$$

for all $0 \le \varepsilon' \le \varepsilon$.

Proof. Using the majorization Lemma 3.3 the proof is basically reduced to a right choice of notation. Let A be the nonnegative compact operator in $L^2(\mathbb{R}, \mathbf{G})$, given by the integral kernel[1] $A(x, y) := W^*(x)W(y)$. Furthermore let

$$g_\varepsilon(dp) = \begin{cases} \varepsilon(\pi(p^2 + \varepsilon^2))^{-1}\, dp & \text{if} \quad \varepsilon > 0 \\ \delta(dp) & \text{if} \quad \varepsilon = 0 \end{cases} \tag{3.3}$$

be the Cauchy distribution and $\{U(p)\}_{p\in\mathbb{R}}$ be the group of unitary multiplication operators $(U(p)\psi)(x) = e^{-ipx}\psi(x)$ on $L^2(\mathbb{R}, \mathbf{G})$. Passing to the Fourier representation of the Green function in (3.2) we obtain

$$\mathcal{L}_\varepsilon = \int_{-\infty}^{\infty} U^*(p) A U(p)\, g_\varepsilon(dp)\,. \tag{3.4}$$

[1] In the scalar case A would just be the rank one operator $|W\rangle\langle W|$ (in Dirac notation).

Of course, $\mathcal{L}_0 = A$. In particular, Lemma 3.3 and (3.4) immediately imply $\mathcal{L}_\varepsilon \prec \mathcal{L}_0$. The Cauchy distribution is a convolution semigroup, i.e. $g_\varepsilon = g_{\varepsilon}{}' * g_{\varepsilon-\varepsilon'}$. If we insert this into (3.4) and change variables using the group property of the unitary operators $U(p)$, then Lemma 3.3 yields

$$\mathcal{L}_\varepsilon = \int U^*(p)\mathcal{L}_{\varepsilon'}U(p)\, g_{\varepsilon-\varepsilon'}(p)\,dp \prec \mathcal{L}_{\varepsilon'}.$$

This completes the proof. ∎

3.3. **Proof of Theorem 3.1.** Let $W(x) = \sqrt{V_-(x)}$, so $W^* = W$. Then from the assumptions made in Theorem 3.1, we find that $W(x)$ is a family of nonnegative Hilbert-Schmidt operators such that $\|W(\cdot)\|_{\mathcal{S}_2} \in L^2(\mathbb{R})$. Let

$$\mathcal{K}_E := \frac{1}{2\sqrt{E}}\mathcal{L}_{\sqrt{E}} = W\left[\left(-\frac{d^2}{dx^2}+E\right)^{-1}\otimes \mathbf{1}_G\right]W\,, \tag{3.5}$$

where $\mathcal{L}_\varepsilon$ is defined in (3.2). According to the Birman-Schwinger principle [4, 24] we have

$$1 = \lambda_j(\mathcal{K}_{E_j})$$

for all negative eigenvalues $\{-E_j\}_j$ of the Schrödinger operator (2.3). Multiplying this equality by $2\sqrt{E_j}$ and summing over j we obtain

$$2\sum\sqrt{E_j} = \sum \lambda_j(\mathcal{L}_{\sqrt{E_j}}). \tag{3.6}$$

In contrast to $\mathcal{K}_E$ the operator $\mathcal{L}_{\sqrt{E}}$ is well-behaved for small energies. We now use the same *monotonicity argument* as in [15] to dispose of the energy dependence of the operator in (3.6). Namely, for any $n \in \mathbb{N}$, Lemma 3.4 implies that the *partial traces* $\sum_{j\le n}\lambda_j(\mathcal{L}_\varepsilon)$ are *monotone decreasing* in ε. Given this monotonicity, a simple induction argument yields

$$\sum_{j\le n}\lambda_j(\mathcal{L}_{\sqrt{E_j}}) \le \sum_{j\le n}\lambda_j(\mathcal{L}_{\sqrt{E_n}}) \quad \text{for all } n\in\mathbb{N}.$$

Hence, by (3.6) we also have the bound

$$2\sum\sqrt{E_j} \le \sum\lambda_j(\mathcal{L}_0) = \operatorname{tr}\mathcal{L}_0 = \int_{-\infty}^{\infty}\operatorname{tr}W^2(x)\,dx = \int_{-\infty}^{\infty}\operatorname{tr}V_-(x)\,dx.$$

The proof is complete.

3.4. **Some generalizations of Theorem 3.1.** The above strategy can be adapted to obtain upper bounds on eigenvalue moments for operators of the form $H = |-i\nabla|^\beta + V$ acting in $L^2(R^d)$, $\beta > d$. Suppose that Φ is an infinitely divisible symmetric probability density, e.g. a compound Poisson, of the form

$$\Phi(p) = \frac{1}{(2\pi)^d}\int e^{\int(\cos(x\cdot\xi)-1)\,dm(\xi)}e^{ip\cdot x}dx$$

with m a non-negative measure and such that Φ satisfies a point-wise inequality

$$\left(|p|^\beta + 1\right)^{-1} \leq c_0 \Phi(p) \tag{3.7}$$

for some constant c_0. Then by scaling,

$$E^{\frac{\beta-d}{\beta}} \left(|p|^\beta + E\right)^{-1} \leq c_0 E^{-d/\beta} \Phi(p/E^\beta) \equiv \Psi_E(p).$$

Moreover,

$$\Psi_E(p) \equiv \Theta_{E,E'} * \Psi_{E'}(p)$$

where $\Theta_{E,E'}$ is a non-negative probability density with Fourier transform given by

$$\hat{\Theta}_{E,E'}(x) = \exp\left\{\int (\cos(x \cdot \xi) - 1)[dm(\xi/E^{1/\beta}) - dm(\xi/{E'}^{1/\beta})]\right\}$$

provided that $[dm(\xi/E^{1/\beta}) - dm(\xi/{E'}^{1/\beta})]$ is non-negative for $E' \leq E$.

Assuming that dm satisfies this condition, we have by the majorization argument that

$$\begin{aligned} \sum_{j \leq n} E_j^{(\beta-d)/\beta}(H) &\leq \sum_{j \leq n} \lambda_j \Big(V_-^{1/2} \frac{E_j^{(\beta-d)/\beta}}{|-i\nabla|^\beta + E_j} V_-^{1/2}\Big) \\ &\leq \sum_{j \leq n} \lambda_j (V_-^{1/2} \Psi_{E_j}(-i\nabla) V_-^{1/2}) \\ &\leq \operatorname{tr}(V_-^{1/2} \Psi_{E_n}(-i\nabla) V_-^{1/2}) = \frac{c_0}{(2\pi)^d} \int V_-(x)\,dx. \end{aligned} \tag{3.8}$$

The problem of finding such an optimal Φ and c_0 seems non-trivial in general. But in d dimensions, with the choice $dm(\xi) = c\,d\xi/|\xi|^{d+\alpha}$, with $d + \alpha \leq \beta$, $0 < \alpha < 2$, $(\cos(x \cdot \xi) - 1)$ is integrable with respect to m, and $\int (\cos(x \cdot \xi) - 1)\,dm(\xi) = -c_1 |x|^\alpha$ for some $c_1 > 0$. Consequently, $\Phi(p) \sim |p|^{-(\alpha+d)}$, $p \to \infty$, and $c_0 \Phi$ will majorize $(|p|^\beta + 1)^{-1}$ for sufficiently large c_0. An eigenvalue moment bound (3.8) follows. For the $d = 1$, $\beta = 2$ Cauchy density case above, the optimal choice is $dm(\xi) = d\xi/(\pi\xi^2)$ and $c_0 = \pi$; (3.7) is an equality.

3.5. **A priori estimate for moments $\gamma \geq 1/2$.** Following Aizenman and Lieb [1] we can "lift" the bound of Theorem 3.1 to moments $\gamma \geq 1/2$.

Corollary 3.5. *Assume that* $V(x)$ *is a nonpositive operator-valued function for a.e.* $x \in \mathbb{R}$ *and that* $\operatorname{tr} V_-(\cdot) \in L^{\gamma+\frac12}(\mathbb{R})$ *for some* $\gamma \geq 1/2$. *Then*

$$\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V\Big)_-^{\gamma} = \sum_j E_j^{\gamma} \leq 2L^{\mathrm{cl}}_{\gamma,1}\int_{-\infty}^{\infty} \operatorname{tr} V_-^{\gamma+\frac12}\,dx\,. \tag{3.9}$$

Proof. Note that Theorem 3.1 is equivalent to

$$\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V\Big)_-^{1/2} \leq 2\iint_{\mathbb{R}\times\mathbb{R}} \operatorname{tr}(p^2 - V_-(x))_-^{1/2}\,\frac{dp\,dx}{2\pi}\,.$$

Scaling gives the simple identity for all $s \in \mathbb{R}$

$$s_-^{\gamma} = C_\gamma \int_0^\infty t^{\gamma-\frac32}(s+t)_-^{1/2}\,dt\,, \qquad C_\gamma^{-1} = B\Big(\gamma-\frac12,\frac32\Big)\,,$$

where B is the Beta function. Let $\mu_j(x)$ the eigenvalues of $V_-(x)$. Then

$$\begin{aligned}
\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V\Big)_-^{\gamma} &= C_\gamma\int_0^\infty dt\, t^{\gamma-\frac32}\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + V + t\Big)_-^{1/2}\\
&\leq C_\gamma \int_0^\infty dt\, t^{\gamma-\frac32}\, 2\iint \operatorname{tr}(p^2 - V_- + t)_-^{1/2}\,\frac{dp\,dx}{2\pi}\\
&= 2\sum_{j=1}^{\infty}\iint\Big[C_\gamma\int_0^\infty dt\, t^{\gamma-\frac32}(p^2-\mu_j+t)_-^{1/2}\Big]\,\frac{dp\,dx}{2\pi}\\
&= 2\iint \operatorname{tr}(p^2 - V_-)_-^{\gamma}\,\frac{dp\,dx}{2\pi} = 2\,L^{\mathrm{cl}}_{\gamma,1}\int \operatorname{tr} V_-^{\gamma+1/2}\,dx\,.
\end{aligned}$$

∎

4. New estimates on the constants $L_{\gamma,d}$ for $1/2 \leq \gamma < 3/2$, $d \in \mathbb{N}$

4.1. **The Main result.** We consider now the Schrödinger operator (2.3) in $L^2(\mathbb{R}^d, \mathbf{G})$ for an arbitrary $d \in \mathbb{N}$. Assume that V is a nonpositive operator-valued function satisfying the condition

$$\operatorname{tr} V(\cdot) \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d) \tag{4.1}$$

for some appropriate γ. We shall discuss bounds on the optimal constants in the Lieb-Thirring inequalities

$$\operatorname{tr}(-\Delta\otimes \mathbf{1} + V)_-^{\gamma} \leq L_{\gamma,d}\int_{\mathbb{R}^d} \operatorname{tr} V_-^{\frac{d}{2}+\gamma}\,dx\,. \tag{4.2}$$

In [17] it has been shown that

$$L_{\gamma,d} = L^{\mathrm{cl}}_{\gamma,d} \qquad \text{for all} \quad \gamma \geq 3/2, \quad d \in \mathbb{N}\,. \tag{4.3}$$

The main result of the paper concerns $1/2 \leq \gamma < 3/2$.

Theorem 4.1. *Let* V *be a nonpositive operator-valued function and let the condition* (4.1) *be satisfied. Then the following estimates on the sharp constants* $L_{\gamma,d}$ *hold*

$$L_{\gamma,d} \leq 2L^{cl}_{\gamma,d} \quad \text{for all} \quad 1 \leq \gamma < 3/2\,, \quad d \in \mathbb{N}\,, \tag{4.4}$$

$$L_{\gamma,d} \leq 2L^{cl}_{\gamma,d} \quad \text{for all} \quad 1/2 \leq \gamma < 3/2\,, \quad d = 1\,, \tag{4.5}$$

$$L_{\gamma,d} \leq 4L^{cl}_{\gamma,d} \quad \text{for all} \quad 1/2 \leq \gamma < 1\,, \quad d \geq 2\,. \tag{4.6}$$

Remark. For the special case $\gamma = 1$ we find that

$$L^{cl}_{1,d} \leq L_{1,d} \leq 2L^{cl}_{1,d} \quad \text{for all} \quad d \in \mathbb{N}\,.$$

Even in the scalar case $\mathbf{G} = \mathbb{C}$ this is a substantial improvement of the previously known numerical estimates on these constants in high dimensions obtained in [5] and [20].

Remark. In fact, our proof of Theorem 4.1 yields

$$L_{\gamma,d} \leq \frac{L_{\gamma,1}}{L^{cl}_{\gamma,1}} L^{cl}_{\gamma,d}, \qquad d \in \mathbb{N}\,, \quad 1 \leq \gamma < 3/2\,.$$

According to Corollary 3.5 we know that $L_{1,1} \leq 2L^{cl}_{1,1}$. In the scalar case Lieb and Thirring conjectured that

$$\frac{L_{\gamma,1}}{L^{cl}_{\gamma,1}} = 2\left(\frac{\gamma - 1/2}{\gamma + 1/2}\right)^{\gamma - 1/2}\,, \qquad 1/2 \leq \gamma < 3/2\,.$$

In particular, if this were true in the matrix case for $\gamma = 1$, our approach would imply $L^{cl}_{1,1} \leq L_{1,d} < 1.16\, L^{cl}_{1,d}$.

Proof of Theorem 4.1. We apply an induction argument similar to the one used in [17]. For $d = 1$ and $1/2 \leq \gamma < 3/2$ the bound (4.5) is identical to (3.9).

Consider the operator (2.3) in the (external) dimension d. We rewrite the quadratic form $h[u,u] + v[u,u]$ for $u \in H^1(\mathbb{R}^d, \mathbf{G})$ as

$$h[u,u] + v[u,u] = \int_{-\infty}^{+\infty} h(x_d)[u,u]\, dx_d + \int_{-\infty}^{+\infty} w(x_d)[u,u]\, dx_d\,,$$

$$h(x_d)[u,u] = \int_{\mathbb{R}^{d-1}} \left\| \frac{\partial u}{\partial x_d} \right\|^2_{\mathbf{G}} dx_1 \cdots x_{d-1}\,,$$

$$w(x_d)[u,u] = \int_{\mathbb{R}^{d-1}} \left[\sum_{j=1}^{d-1} \left\| \frac{\partial u}{\partial x_j} \right\|^2_{\mathbf{G}} + \langle V(x)u, u\rangle_{\mathbf{G}} \right] dx_1 \cdots x_{d-1}\,.$$

The form $w(x_d)$ is closed on $H^1(\mathbb{R}^{d-1}, \mathbf{G})$ for a.e. $x_d \in \mathbb{R}$ and it induces the self-adjoint operator

$$W(x_d) = -\sum_{k=1}^{d-1} \frac{\partial^2}{\partial x_k^2} \otimes \mathbf{1}_{\mathbf{G}} + V(x_1, \dots, x_{d-1}; x_d)$$

on $L^2(\mathbb{R}^{d-1}, \mathbf{G})$. For a fixed $x_d \in \mathbb{R}$ this is a Schrödinger operator in $d-1$ dimensions. Its negative spectrum is discrete, hence $W_-(x_d)$ is compact on $L^2(\mathbb{R}^{d-1}, \mathbf{G})$.

Assume that we have (4.4)–(4.5) for the dimension $d-1$ and all γ from the interval $1/2 \le \gamma < 3/2$. Then $\operatorname{tr} W_-^{\gamma+\frac{1}{2}}(x_d)$ satisfies the bound

$$\operatorname{tr} W_-^{\gamma+\frac{1}{2}}(x_d) \le L_{\gamma+\frac{1}{2},d-1} \int_{\mathbb{R}^{d-1}} \operatorname{tr} V_-^{\gamma+\frac{d}{2}}(x_1, \dots, x_{d-1}; x_d)\, dx_1 \cdots dx_{d-1} \tag{4.7}$$

for a.e. $x_d \in \mathbb{R}$. Here

$$L_{\gamma+\frac{1}{2},d-1} = L^{\mathrm{cl}}_{\gamma+\frac{1}{2},d-1} \qquad \text{for} \qquad \gamma \ge 1\,, \tag{4.8}$$

$$L_{\gamma+\frac{1}{2},d-1} \le 2L^{\mathrm{cl}}_{\gamma+\frac{1}{2},d-1} \qquad \text{for} \qquad 1/2 \le \gamma < 1\,. \tag{4.9}$$

Indeed, (4.8) follows from (4.3) and (4.9) follows from (4.4)–(4.5) in dimension $d-1$.

Let $w_-(x_d)[\cdot,\cdot]$ be the quadratic form corresponding to the operator $W_-(x_d)$ on $\mathbf{H} = L^2(\mathbb{R}^{d-1}, \mathbf{G})$. We have $w(x_d)[u,u] \ge -w_-(x_d)[u,u]$ and

$$h[u,u] + v[u,u] \ge \int_{-\infty}^{+\infty} \left[\left\| \frac{\partial u}{\partial x_d} \right\|_{\mathbf{H}}^2 - \langle W_-(x_d) u, u \rangle_{\mathbf{H}} \right] dx_d \tag{4.10}$$

for all $u \in H^1(\mathbb{R}^d, \mathbf{G})$. According to section 2.2 the form on the r.h.s. of (4.10) can be closed to $H^1(\mathbb{R}, \mathbf{H})$ and induces the self-adjoint operator

$$-\frac{d^2}{dx_d^2} \otimes \mathbf{1}_{\mathbf{H}} - W_-(x_d)$$

on $L^2(\mathbb{R}, \mathbf{H})$. Then (4.10) implies

$$\operatorname{tr}(-\Delta \otimes \mathbf{1}_{\mathbf{G}} + V)_-^{\gamma} \le \operatorname{tr} \left(-\frac{d^2}{dx_d^2} \otimes \mathbf{1}_{\mathbf{H}} - W_-(x_d) \right)_-^{\gamma} . \tag{4.11}$$

The assumption $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$ implies that $\operatorname{tr} W_-^{\gamma+\frac{1}{2}}$ is an integrable function and we can apply Corollary 3.5 to the r.h.s. of (4.11). In view of (4.7) we find

$$\operatorname{tr}\left(-\frac{d^2}{dx_d^2}\otimes \mathbf{1}_{\mathbf{H}} - W_-(x_d)\right)_-^\gamma \le L_{\gamma,1}\int_{-\infty}^{+\infty} \operatorname{tr} W_-^{\gamma+\frac12}(x_d)\,dx_d$$

$$\le L_{\gamma,1}L_{\gamma+\frac12,d-1}\int_{\mathbb{R}^d} \operatorname{tr} V_-^{\gamma+\frac d2}\,dx$$

for $\gamma \ge 1/2$. The bounds (4.5), (4.8) or (4.9) and the calculation

$$L^{cl}_{\gamma,1}L^{cl}_{\gamma+\frac12,d-1} = \frac{\Gamma(\gamma+1)}{2\pi^{\frac12}\Gamma(\gamma+\frac12+1)}\cdot\frac{\Gamma(\gamma+\frac12+1)}{2^{d-1}\pi^{\frac{d-1}{2}}\Gamma(\gamma+\frac12+\frac{d-1}{2}+1)}$$

$$= \frac{\Gamma(\gamma+1)}{2^d\pi^{\frac d2}\Gamma(\gamma+\frac d2+1)} = L^{cl}_{\gamma,d}$$

complete the proof. ∎

4.2. **Estimates for magnetic Schrödinger operators.** Following a remark by B. Helffer [13] and using the arguments from [17] we can extend Theorem 4.1 to Schrödinger operators with magnetic fields. Let $Q(\mathbf{a})$ be a self-adjoint operator in $L^2(\mathbb{R}^d, \mathbf{G})$

(4.12) $$Q(\mathbf{a}) = (i\nabla + \mathbf{a}(x))^2 \otimes \mathbf{1}_{\mathbf{G}} + V(x),$$

where

$$\mathbf{a}(x) = (a_1(x), \cdots, a_d(x))^t, \qquad d \ge 2,$$

is a magnetic vector potential with real-valued entries $a_k \in L^2_{loc}(\mathbb{R}^d)$.

We consider the inequality

(4.13) $$\operatorname{tr}(Q(\mathbf{a}))_-^\gamma \le \tilde{L}_{\gamma,d}\int_{\mathbb{R}^d} V_-^{\frac d2+\gamma}\,dx,$$

where the nonpositive operator function $V(\cdot)$ satisfies (4.1). In [17] it has been shown, that

(4.14) $$\tilde{L}_{\gamma,d} = L^{cl}_{\gamma,d} \qquad \text{for all} \qquad \gamma \ge 3/2, \quad d \in \mathbb{N}.$$

In general, the sharp constant $\tilde{L}_{\gamma,d}$ in (4.14) might differ from the sharp constant $L_{\gamma,d}$ in (4.2)

$$L^{cl}_{\gamma,d} \le L_{\gamma,d} \le \tilde{L}_{\gamma,d}.$$

By combining the arguments from [17] and those used in the prove of Theorem 4.1 we immediately obtain the following result:

Theorem 4.2. *The following estimates on the sharp constants* $\tilde{L}_{\gamma,d}$ *in* (4.13) *hold*

$$\tilde{L}_{\gamma,d} \le 2L^{cl}_{\gamma,d} \qquad \textit{for all} \qquad 1 \le \gamma < 3/2\,, \quad d \ge 2\,, \tag{4.15}$$

$$\tilde{L}_{\gamma,d} \le 4L^{cl}_{\gamma,d} \qquad \textit{for all} \quad 1/2 \le \gamma < 1 \quad , \quad d \ge 2\,. \tag{4.16}$$

5. Trace formulae and estimates from below for $d = 1$.

5.1. **Matrix-valued potentials.** Let $\mathbf{G} = \mathbb{C}^n$ be a finite dimensional Hilbert space. We consider the system of ordinary differential equations

$$-\left(\frac{d^2}{dx^2} \otimes \mathbf{1}\right) y(x) + V(x)y(x) = k^2 y(x)\,, \qquad x \in \mathbb{R}\,, \tag{5.1}$$

where V is a compactly supported, smooth (not necessary sign definite) Hermitian matrix-valued function. Define

$$x_{\min} := \min \operatorname{supp} V \quad \text{and} \quad x_{\max} := \max \operatorname{supp} V\,.$$

Then for any $k \in \mathbb{C}\setminus\{0\}$ there exist unique $n \times n$ matrix-solutions $F(x,k)$ and $G(x,k)$ of the equations

$$-F''_{xx}(x,k) + VF(x,k) = k^2 F(x,k)\,, \tag{5.2}$$

$$-G''_{xx}(x,k) + VG(x,k) = k^2 G(x,k)\,, \tag{5.3}$$

satisfying

$$F(x,k) = e^{ikx}\mathbf{1}_{\mathbf{G}} \quad \text{as} \quad x \ge x_{\max}\,,$$
$$G(x,k) = e^{-ikx}\mathbf{1}_{\mathbf{G}} \quad \text{as} \quad x \le x_{\min}\,.$$

If $k \in \mathbb{C} \setminus \{0\}$, then the pairs of matrices $F(x,k)$, $F(x,-k)$ and $G(x,k)$, $G(x,-k)$ form full systems of independent solutions of (5.1). Hence the matrix $F(x,k)$ can be expressed as a linear combination of $G(x,k)$ and $G(x,-k)$

$$F(x,k) = G(x,k)B(k) + G(x,-k)A(k)\,. \tag{5.4}$$

The matrix functions $A(k)$ and $B(k)$ are uniquely defined by (5.4).

5.2. **Trace formulae.** In [17] the Buslaev-Faddeev-Zakharov trace formulae were generalized for the matrix-valued potentials satisfying the conditions from the previous subsection. We recall here the first three trace identities given by the equations (1.60)-(1.62) from [17]

$$\frac{1}{4}\int_{-\infty}^{+\infty} \operatorname{tr} V\,dx = I_0 - \sum_{l=1}^{N} E_l^{1/2}, \tag{5.5}$$

$$\frac{3}{16}\int_{-\infty}^{+\infty} \operatorname{tr} V^2\,dx = 3I_2 + \sum_{l=1}^{N} E_l^{3/2}, \tag{5.6}$$

$$\frac{5}{32}\int_{-\infty}^{+\infty} \operatorname{tr} V^3\,dx + \frac{5}{64}\int_{-\infty}^{+\infty} \operatorname{tr}\left(\frac{dV}{dx}\right)^2 dx = 5I_4 - \sum_{l=1}^{N} E_l^{5/2}, \tag{5.7}$$

where

$$I_j = (2\pi)^{-1}\int_{-\infty}^{+\infty} k^j \ln|\det A(k)|\,dk \qquad j = 0, 2, 4\,.$$

Note that for real k's we have (cf. (1.11) in [17])

$$A(k)A^*(k) = \mathbf{1}_{\mathbf{G}} + B(-k)B^*(-k)$$

Thus we obtain $|\det A(k)| \geq 1$ for all $k \in \mathbb{R}$ and

$$I_j \geq 0 \qquad j = 0, 2, 4\,. \tag{5.8}$$

Remark. Notice that

$$L^{cl}_{1/2,1} = 1/4\,, \qquad L^{cl}_{3/2,1} = 3/16\,, \qquad L^{cl}_{5/2,1} = 5/32\,. \tag{5.9}$$

5.3. $\gamma = 1/2$. The identity (5.5) immediately leads to a bound from below on the sum of the square roots of the operator (5.1). Indeed, (5.8) implies

$$L^{cl}_{1/2,1}\int (\operatorname{tr} V_- - \operatorname{tr} V_+)\,dx \leq \sum_l E_l^{1/2}. \tag{5.10}$$

For the scalar case this estimate has been pointed out in [11], see also [25]. By continuity this bound extends to all matrix functions V, for which

$$\operatorname{tr} V_+(\cdot) \in L^1(\mathbb{R}) \qquad \text{and} \qquad \operatorname{tr} V_-(\cdot) \in L^1(\mathbb{R})\,. \tag{5.11}$$

Using a standard density argument and (3.1) we conclude, that (5.10) holds also for general separable Hilbert spaces $\mathbf{G}$. This implies

Corollary 5.1. *Let* $V(x) \in S_1(\mathbf{G})$ *and* $\operatorname{tr} V_\pm(\cdot) \in L_1(\mathbb{R})$. *Then for the 1/2 moments of the negative eigenvalues of the operator* (2.3) *we have the following two side inequalities*

$$L^{cl}_{1/2,1}\int (\operatorname{tr} V_- - \operatorname{tr} V_+)\,dx \leq \sum_l E_l^{1/2} \leq 2L^{cl}_{1/2,1}\int \operatorname{tr} V_-\,dx.$$

5.4. $\boldsymbol{\gamma = 3/2}$**.** Let us return to the case $\mathbf{G} = \mathbb{C}^n$ and let V be a smooth, compactly supported matrix-valued function. The upper bound (3.1) and the identity (5.5) imply

$$
\begin{aligned}
I_1 &= \sum_{l=1}^{N} E_l^{1/2} + L^{cl}_{1/2,1} \int (\operatorname{tr} V_+ - \operatorname{tr} V_-)\, dx \\
&\le L^{cl}_{1/2,1} \int (\operatorname{tr} V_+ + \operatorname{tr} V_-)\, dx\,.
\end{aligned} \tag{5.12}
$$

Moreover, from (4.3) with $d = 1$ and $\gamma = 5/2$, (5.7) and (5.9) it follows that

$$
\begin{aligned}
5I_4 &= L^{cl}_{5/2,1} \int (\operatorname{tr} V_+^3 - \operatorname{tr} V_-^3)\, dx + \frac{1}{2} L^{cl}_{5/2,1} \int \operatorname{tr} \left(\frac{dV}{dx} \right)^2 dx + \sum_l E_l^{5/2} \\
&\le L^{cl}_{5/2,1} \int \operatorname{tr} V_+^3\, dx + \frac{1}{2} L^{cl}_{5/2,1} \int \operatorname{tr} \left(\frac{dV}{dx} \right)^2 dx\,.
\end{aligned} \tag{5.13}
$$

Note that in the scalar case the inequalities (5.12) and (5.13) with somewhat worse constant were found in [25] and [21] respectively. These estimates together with Hölder's inequality give

$$
I_2 \le I_0^{1/2} I_4^{1/2} = \frac{1}{16} \left[\int (\operatorname{tr} V_+ + \operatorname{tr} V_-)\, dx \right]^{\frac12} \left[2 \int \operatorname{tr} V_+^3\, dx + \int \operatorname{tr} \left(\frac{dV}{dx} \right)^2 dx \right]^{\frac12}. \tag{5.14}
$$

Inserting (5.14) into (5.6) and considering the special case $V_+ = 0$, we find

$$
\frac{3}{16} \int \operatorname{tr} V_-^2\, dx - \sum_l E_l^{3/2} \le \frac{3}{16} \left[\int \operatorname{tr} V_-\, dx \right]^{\frac12} \left[\int \operatorname{tr} \left(\frac{dV}{dx} \right)^2 dx \right]^{\frac12}. \tag{5.15}
$$

Standard density and continuity arguments allow us to extend (5.15) to general separable Hilbert spaces $\mathbf{G}$ and arbitrary nonpositive operator-valued potentials V, for which all integrals in (5.15) are finite.

5.5. **A remainder term.** Let us discuss further the inequality (5.15). First note, that in view of (4.3) for $d = 1$ and $\gamma = 3/2$, the l.h.s. of (5.15) is nonnegative. Therefore the inequalities (5.15) can be interpreted as an estimate on the difference between the sum $\sum_l E_l^{3/2}$ and the classical phase space integral

$$
L^{cl}_{3/2} \int \operatorname{tr} V_-^2 dx = \iint \operatorname{tr}(p^2 + V(x))_-^{3/2} \frac{dp\, dx}{2\pi}\,.
$$

By replacing V by αV we obtain the following result:

Theorem 5.2. *Assume that* V *is a nonpositive operator-valued function such that* $\operatorname{tr} V_- \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $\operatorname{tr}(dV/dx)^2 \in L^1(\mathbb{R})$*. Then*

$$\operatorname{tr}\Big(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + \alpha V\Big)_-^{3/2} = \alpha^2 L^{\mathrm{cl}}_{3/2,1}\int \operatorname{tr} V_-^2\, dx - R(\alpha)$$

for all $\alpha > 0$*, where*

$$0 \le R(\alpha) \le \frac{3\alpha^{3/2}}{16}\left[\int \operatorname{tr} V_-\, dx\right]^{\frac{1}{2}} \left[\int \operatorname{tr}\left(\frac{dV}{dx}\right)^2 dx\right]^{\frac{1}{2}}.$$

Remark. For large values of the coupling constant α, Theorem 5.2 gives us the correct order $O(\alpha^{3/2})$ of the remainder term in the Weyl asymptotic formula for $3/2$-moments of the negative eigenvalues.

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