ON SHARP STRICHARTZ INEQUALITIES IN LOW DIMENSIONS

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Abstract. Recently Foschi gave a proof of a sharp Strichartz inequality in one and two dimensions. In this note, a new representation in terms of an orthogonal projection operator is obtained for the space time norm of solutions of the free Schrödinger equation in dimension one and two. As a consequence, the sharp Strichartz inequality follows from the elementary property that orthogonal projections do not increase the norm.

1. Introduction

The solution u to the free Schrödinger equation

$$i\partial_t u = -\Delta u \quad \text{on } \mathbb{R}^d \tag{1.1}$$

with initial condition $u(0) = f \in L^2(\mathbb{R}^2)$ is given by

$$u(t,x) = (\mathrm{e}^{it\Delta} f)(x) \tag{1.2}$$

where $\mathrm{e}^{it\Delta}$ is defined, for example, by the spectral theorem. Since $\mathrm{e}^{it\Delta}$ is a unitary group on $L^2(\mathbb{R}^d)$, one immediately sees that $u \in L^\infty_t(L^2(\mathbb{R}^d))$. But in fact, due to the dispersive nature of the free Schrödinger equation, the solution u, as a function of space-time, is even L^p-bounded,

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^d)} \le S_d \|f\|_{L^2(\mathbb{R}^d)} \tag{1.3}$$

where $p = p(d) = 2 + \frac{4}{d}$. This was first shown by Strichartz [10] who followed the L^p restriction proof of Stein-Tomas. Later simplified proofs were given by Ginibre and Velo [4], see also [2, 11].

The sharp value of S_d, i.e., the quantity

$$S_d = \sup_{f\neq 0,} \frac{\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}} \tag{1.4}$$

has been unknown until very recently. In fact, even the existence of maximizers for (1.4), that is, functions $f_* \neq 0$ such that one has equality in (1.4),

$$S_d = \frac{\|\mathrm{e}^{it\Delta} f_*\|_{L^p(\mathbb{R}^{d+1})}}{\|f_*\|_{L^2(\mathbb{R})}} \tag{1.5}$$

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has been only recently established. By using an elaborate application of the concentration compactness method, Markus Kunze showed in [5] that (1.4) has a maximizer in one dimension. His proof does not, however, provide any explicit information about the maximizer nor the value of S_1. The reason why even the existence of maximizers has not been known until recently is the invariance of the Strichartz inequality under the rather large group of Galilei transformations and scaling. This makes the usual existence proof for maximizers via minimizing sequences very hard, since they can very easily converge weakly to zero. The usual method to circumvent this problem is the concentration compactness principle, however, in this setting it has to be used twice, first in Fourier space, then in real space, see [5] for the Strichartz inequality and [6] for the maximizer in the dispersion management soliton equation, which is related to the Strichartz inequality. However, Kunze's approach gives no information about the maximizer, not even smoothness. For the dispersion management problem, Stanislavlova showed recently in [9] that Kunze's maximizer is, indeed, given by a smooth function.

Very recently, Damiano Foschi [3] gave a simple proof of the Strichartz inequality in one and two dimensions, which yields the sharp constant. More precisely, he proved

Theorem 1.1 (Foschi, [3])**.** *The sharp constants for the Strichartz inequality in one and two dimensions are* $S_1 = 12^{-1/12}$ *and* $S_2 = 2^{-1/2}$*, respectively. Moreover, one has equality in the Strichartz estimate in dimension one and two if the initial condition* f *is given by a Gaussian.*

Remark 1.2. In the first version of his preprint, Foschi proved only that Gaussians are among the maximizers in the Strichartz inequality. The existence of non-Gaussian maximizers was not ruled out. While preparing our manuscript, we noticed that Foschi replaced the old version of his preprint by a new version, in which he also characterizes the maximizers. We believe that our approach is technically simpler than Foschi's, especially our way to characterize the maximizers, and it sheds some different light on the Strichartz inequality.

The main purpose of this note is to give a simple new representation of the Strichartz integral, see Theorem 1.3 below. It shows that the Strichartz estimate follows from an upper bound on a *very simple linear* operator and, moreover, gives a geometric criterion for the maximizer in the Strichartz inequality. For $x, y \in \mathbb{R}^d$ denote by $x \cdot y$ the Euclidean scalar product. For $f, g \in L^2(\mathbb{R}^d)$, $f \otimes g$ is the usual tensor product, $\mathbb{R}^d \times \mathbb{R}^d \ni x = (x_1, x_2) \to f \otimes g(x) := f(x_1)g(x_2)$. Similarly for the triple tensor product $f_1 \otimes f_2 \otimes f_3$. We denote by

$$L_1 = \text{closure}\big(\text{span}\{\widetilde{G}((1,1,1)\cdot\eta, |\eta|^2) |\, \widetilde{G} \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}_+)\}\big) \tag{1.6}$$

the closed linear subspace of $L^2(\mathbb{R}^3)$ consisting of functions invariant under rotations of $\mathbb{R}^3$ which keep the $(1,1,1)$ direction fixed and by $P_1 : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be the orthogonal projection operator onto L_1.

Moreover, let $\alpha_1 = (1,0,1,0)$, $\alpha_2 = (0,1,0,1)$ and

$$L_2 = \text{closure}\big(\text{span}\{\widetilde{G}(\alpha_1\cdot\eta, \alpha_2\cdot\eta, |\eta|^2) |\, \widetilde{G} \in \mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)\}\big) \tag{1.7}$$

be the closed linear subspace of $L^2(\mathbb{R}^4)$ consisting of functions invariant under rotations of $\mathbb{R}^4$ which keep both the $(1,0,1,0)$ and $(0,1,0,1)$ directions fixed. Define $P_2 : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$ to be the orthogonal operator onto functions L_2. With this, we have

Theorem 1.3 (Representation Theorem)**.** *Let* $f \in L^2(\mathbb{R}^d)$.
a) If $d = 1$ *then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |e^{it\Delta} f(x)|^6 \, dxdt = \frac{1}{2\sqrt{3}} \langle f\otimes f\otimes f, P_1(f\otimes f\otimes f)\rangle_{L^2(\mathbb{R}^3)}$$

b) If $d = 2$ *then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} |e^{it\Delta} f(x)|^4 \, dxdt = \frac{1}{4} \langle f\otimes f, P_2(f\otimes f)\rangle_{L^2(\mathbb{R}^4)}.$$

Remark 1.4. We would like to stress that we are not using the Fourier transformation in our discussion of the Strichartz inequality. Nevertheless, using the Fourier transform to represent the solution u of the free Schrödinger equation in the form

$$u(t,x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot k} e^{-it|k|^2} \hat{f}(k)\, dk,$$

instead of formula (2.1) below, an argument very much similar to the one given in section 2 shows that in dimension one

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |e^{it\Delta} f(x)|^6 \, dxdt = \frac{1}{2\sqrt{3}} \langle \hat{f}\otimes \hat{f}\otimes \hat{f}, P_1(\hat{f}\otimes \hat{f}\otimes \hat{f})\rangle_{L^2(\mathbb{R}^3)}$$

and in dimension two

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} |e^{it\Delta} f(x)|^4 \, dxdt = \frac{1}{4} \langle \hat{f}\otimes \hat{f}, P_2(\hat{f}\otimes \hat{f})\rangle_{L^2(\mathbb{R}^4)}.$$

Thus there seems to be a symmetry of the Strichartz norm under Fourier transformation. We do not use this additional symmetry, but it might be useful in other contexts.

Using that any projection operator is bounded above by the identity, the Representation Theorem 1.3 immediately yields the sharp Strichartz inequality: In one dimension

$$\|e^{it\Delta} f\|^6_{L^6(\mathbb{R})} \le \frac{1}{2\sqrt{3}} \langle f\otimes f\otimes f, f\otimes f\otimes f\rangle_{L^2(\mathbb{R}^3)} = \frac{1}{2\sqrt{3}} \|f\|^6_{L^2(\mathbb{R})} \tag{1.8}$$

Hence we get the bounds $S_1 \le (1/(2\sqrt{3}))^{1/6} = 12^{-1/12}$. Similarly, for the two dimensional case, $S_2 \le 4^{-1/4} = 2^{-1/2}$. One also sees that, in order to have equality in the one-dimensional Strichartz inequality, the function $f\otimes f\otimes f$ must be in the range of P_1, that is, it must be invariant under rotations of $\mathbb{R}^3$ which keep the $(1,1,1)$ direction fixed. Obviously, any function of the form $f(x) = e^{-x^2}$ satisfies this condition, since the resulting product is even invariant under arbitrary rotations of $\mathbb{R}^3$. More generally, one can also allow for a linear term in the exponential. That is, all functions of the form

$$f(x) = A\mathrm{e}^{(-\lambda+i\mu)x^2+cx},$$

with $\lambda > 0, \mu \in \mathbb{R}, c \in \mathbb{C}, A \in \mathbb{C}$, give equality in the Strichartz inequality. In particular, the sharp constant is given by $S_1 = 12^{-1/12}$.

Similarly, $f \in L^2(\mathbb{R}^2)$ is a maximizer for the two-dimensional Strichartz inequality, if and only if $f \otimes f$ is in the range of P_2. That is, it is invariant under rotations of $\mathbb{R}^4$ keeping the $(1,0,1,0)$ and the $(0,1,0,1)$ directions fixed. Again, any centered Gaussian (this time with rotational symmetric covariance) and, moreover, all functions of the form

$$\mathbb{R}^2 \ni x \mapsto f(x) = A\mathrm{e}^{(-\lambda+i\mu)|x-x_0|^2+c\cdot x},$$

with $A \in \mathbb{C}$, $c \in \mathbb{C}^2$, and $\lambda > 0$ and $\mu \in \mathbb{R}$ (to ensure integrability of f), will satisfy this condition. So the sharp constant is indeed $S_2 = 2^{-1/2}$.

In fact, the above invariance under suitable rotations together with a simple argument shows that at least in one and two dimensions the *only maximizers* in the Strichartz inequality are Gaussians. More precisely, we have the following

Theorem 1.5 (Gaussian maximizers)**.** *Let $d = 1$ or 2. The function $f \in L^2(\mathbb{R}^d)$ is a maximizer for the Strichartz inequality* (1.3)*,*

$$S_d = \frac{\|\mathrm{e}^{it\Delta} f\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}},$$

if and only if f is a Gaussian, that is,

$$f(x) = A\mathrm{e}^{(-\lambda+i\mu)|x-a|^2+b\cdot x}.$$

where $A \in \mathbb{C}$, $\lambda > 0$, $\mu \in \mathbb{R}$, $a \in \mathbb{R}^d$, and $b \in \mathbb{C}^d$.

Remarks 1.6. (i) The above theorem says, in particular, that the extremizer in two dimensions is the product of two one-dimensional Gaussians.
(ii) We cannot say anything about the three and higher dimensional case, except that Gaussians are solutions of the corresponding Euler-Lagrange equation, hence extremizers, see section 4 We expect the maximizers to be products of one dimensional Gaussians. In fact, following Lieb's ideas [7], see also [1], showing that any maximizer is a product of one dimensional functions would immediately imply that every maximizer is a Gaussian.

Conjecture: Based on Foschi's result and on the direct calculation of the ratio defining S_d for Gaussians, we conjecture

$$S_d = \Big(\frac{1}{2}\Big)^{1/p}\Big(\frac{2}{p}\Big)^{2/(p(p-2))} = \Big(\frac{1}{2}\Big(\frac{d}{d+2}\Big)^{d/2}\Big)^{d/(2d+4)}. \tag{1.9}$$

In the next section, we give the proof of the Representation Theorem 1.3 and in section 3 we give the proof of Theorem 1.5.

2. Proof of the Representation Theorem

If f is an integrable function on $\mathbb{R}^d$, $u(t,x) = e^{it\Delta} f(x)$ is given by

$$u(t,x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{(x-y)^2}{4t}} f(y)\, dy. \tag{2.1}$$

2.1. **One dimensional case.** Now, first consider the case of one space dimension. We want to rewrite $\int_{\mathbb{R}\times\mathbb{R}} |u(t,x)|^6\, dtdx$ using (2.1). With the notation $\eta = (\eta_1, \eta_2, \eta_3), \zeta = (\zeta_1, \zeta_2, \zeta_3)$, and $f \otimes f \otimes f(\eta) := f(\eta_1)f(\eta_2)f(\eta_3)$ and $(1,1,1)\cdot\eta = \eta_1 + \eta_2 + \eta_3$, one gets

$$|u(t,x)|^6 = \frac{1}{(4\pi t)^3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-i\frac{x(1,1,1)\cdot(\eta-\zeta)}{2t}} e^{i\frac{|\eta|^2 - |\zeta|^2}{4t}} \overline{f\otimes f \otimes f(\eta)} f\otimes f\otimes f(\zeta)\, d\eta d\zeta.$$

First integrate this with respect to x, making the substitution $x = 2tz$, and then integrate with respect to t, making the substitution $t = \frac{1}{4\tau}$, and using that, as distributions, $\delta(\xi) = (2\pi)^{-1} \int_{\mathbb{R}} \mathrm{e}^{-it\xi}\, dt$ one arrives at

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} &|u(t,x)|^6\, dxdt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \delta((1,1,1)\cdot(\eta-\zeta))\delta(|\eta|^2 - |\zeta|^2) \overline{f\otimes f\otimes f(\eta)} f\otimes f\otimes f(\zeta)\, d\eta d\zeta\end{aligned} \tag{2.2}$$

We want to identify the right hand side of (2.2) as a quadratic form of a simple self-adjoint operator. Let $F, G \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, the space of test functions and define

$$Q_1(F,G) := \frac{1}{2\pi} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \delta((1,1,1)\cdot(\eta-\zeta))\delta(|\eta|^2 - |\zeta|^2)\overline{F(\eta)}G(\zeta)\, d\eta d\zeta. \tag{2.3}$$

This is a symmetric quadratic form and, by definition,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |u(t,x)|^6\, dxdt = Q_1(f\otimes f\otimes f, f\otimes f\otimes f).$$

The crucial observation is that there should be a symmetric linear operator A_1 generating this form, $Q_1(F,G) = \langle F, A_1 G\rangle_{L^2(\mathbb{R}^3)}$. From (2.3) we can read off A_1,

$$(A_1 G)(\eta) := \frac{1}{2\pi} \int_{\mathbb{R}^3} G(\zeta)\, \delta((1,1,1)\cdot(\eta-\zeta))\delta(|\eta|^2 - |\zeta|^2)\, d\zeta \tag{2.4}$$

First, we have to show that A_1 is, indeed, a bounded operator. It is certainly defined on $\mathcal{C}_0^\infty(\mathbb{R}^3)$, the space of test functions.

Lemma 2.1. *The operator A_1 defined in* (2.4) *maps $\mathcal{C}_0^\infty(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ and extends to a bounded linear operator on all of $L^2(\mathbb{R}^3)$.*

To prove this Lemma, we need an extension of a result contained in [3],

Lemma 2.2. (i) *For all $\eta \in \mathbb{R}^3$, the measure*

$$m_{1,\eta}(d\zeta) = \tfrac{\sqrt{3}}{\pi}\delta((1,1,1)\cdot(\eta-\zeta))\delta(|\eta|^2 - |\zeta|^2)\, d\zeta$$

is a probability measure on $\mathbb{R}^3$.

(ii) *For all Borel measurable sets $B \subset \mathbb{R}^3$, we have*

$$\int_{\mathbb{R}^3} m_{1,\eta}(B)\, d\eta = |B|$$

where $|B|$ denotes the Lebesgue measure of B.

Proof. The proof of the first part of the lemma is a simple calculation: Let $\tau = (1,1,1)\cdot\eta$, $\xi = |\eta|^2$, and $\gamma = \xi - \tau^2/3 = |\eta|^2 - ((1,1,1)\cdot\eta)^2/3 \geq 0$. Rotating the $(1,1,1)$ direction to the north pole, the ζ_1 direction, one sees

$$\begin{aligned} m_{1,\eta}(\mathbb{R}^3) &= \frac{\sqrt{3}}{\pi}\int_{\mathbb{R}^3} \delta(\xi - |\zeta|^2)\delta(\tau - (1,1,1)\cdot\zeta)\, d\zeta \\ &= \frac{1}{\pi}\int_{\mathbb{R}^3} \delta(\xi - |\zeta|^2)\delta(\tau - \sqrt{3}\zeta_1)\, \sqrt{3} d\zeta_1 d(\zeta_2,\zeta_3) \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} \delta(\xi - \frac{\tau^2}{3} - |(\zeta_2,\zeta_3)|^2) d(\zeta_2,\zeta_3) \\ &= 2\int_0^\infty \delta(\gamma - r^2)\, r\, dr = \int_0^\infty \delta(\gamma - s)\, ds = 1. \end{aligned}$$

To prove the second part, note that by Fubini-Tonelli, the symmetric dependence of m_1 on η and ζ, and the first part of the Lemma, we have

$$\begin{aligned} \int_{\mathbb{R}^3} m_{1,\eta}(B)\, d\eta &= \frac{\sqrt{3}}{\pi}\int_{\mathbb{R}^3}\int_B \delta((1,1,1)\cdot(\eta - \zeta))\delta(|\eta|^2 - |\zeta|^2)\, d\zeta d\eta \\ &= \frac{\sqrt{3}}{\pi}\int_B\int_{\mathbb{R}^3} \delta((1,1,1)\cdot(\zeta - \eta))\delta(|\zeta|^2 - |\eta|^2)\, d\eta d\zeta \\ &= \int_B m_{1,\zeta}(\mathbb{R}^3)\, d\zeta = \int_B d\zeta = |B|. \end{aligned}$$

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Proof of Lemma 2.1. Take $G \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. With the measure $m_{1,\eta}$ defined above, the operator A_1 applied to G can be written as

$$(A_1 G)(\eta) = \frac{1}{2\sqrt{3}}\int_{\mathbb{R}^3} G(\zeta)\, m_{1,\eta}(d\zeta).$$

Thus

$$\begin{aligned} \|A_1 G\|^2_{L^2(\mathbb{R}^3)} = \langle A_1 G, A_1 G\rangle_{L^2(\mathbb{R}^3)} &= \frac{1}{12}\int_{\mathbb{R}^3} \Big|\int_{\mathbb{R}^3} G(\zeta)\, m_{1,\eta}(d\zeta)\Big|^2 d\eta \\ &\leq \frac{1}{12}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |G(\zeta)|^2\, m_{1,\eta}(d\zeta)\, d\eta \\ &= \frac{1}{12}\int_{\mathbb{R}^3} |G(\zeta)|^2\, d\zeta = \frac{1}{12}\|G\|^2_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where, for the inequality, we used Jensen's inequality, [8], and the fact that $m_{1,\eta}$ is a probability measure for all η by the first part of Lemma 2.2. The second to last equality follows from interchanging the integrations and then using $\int_{\mathbb{R}^3} m_{1,\eta}(d\zeta)\, d\eta = d\zeta$, which holds thanks to the second part of Lemma 2.2.

Since A_1 is densely defined, the above calculation shows that it extends to a bounded symmetric operator on all of $L^2(\mathbb{R}^3)$ with norm at most $1/(2\sqrt{3})$. ∎

Now we proceed with the proof of the Representation Theorem in the one-dimensional case. By a slight but common abuse of notation we denote by A_1 the extension of A_1 to all of $L^2(\mathbb{R}^3)$. We have to show that A_1 is a multiple of the orthogonal projection operator P_1.

Let $G \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. Note that, by construction, $A_1 G$ depends on η only via $|\eta|^2$ and the projection of η on the line given by the $(1,1,1)$ direction. In other words,

$$(A_1 G)(\eta) = \widetilde{G}((1,1,1)\cdot\eta, |\eta|^2),$$

for some function $\widetilde{G}$ on $\mathbb{R}\times\mathbb{R}_+$. Hence recalling the definition (1.6) for L_1, we see that A_1 maps $\mathcal{C}_0^\infty(\mathbb{R}^3)$ into L_1, that is, into the range of P_1. By denseness of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3)$ and boundedness of A_1, it also maps all of $L^2(\mathbb{R}^3)$ into L_1. And so by symmetry of A_1, $A_1(L_1^\perp) = \{0\}$. To finish the proof of part a) of Theorem 1.3, it remains to check that A_1 acts as a multiple of the identity in the range of P_1.

Take $\widetilde{G} \in \mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{R}_+)$. Then, by the above discussion, the function $G(\zeta) = \widetilde{G}((1,1,1)\cdot\zeta, |\zeta|^2) \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ is in the range of P_1 and

$$\begin{aligned} 2\sqrt{3}(A_1 G)(\eta) &= \frac{\sqrt{3}}{\pi}\int_{\mathbb{R}^3} \widetilde{G}((1,1,1)\cdot\zeta, |\zeta|^2)\delta((1,1,1)\cdot(\eta-\zeta))\delta(|\eta|^2-|\zeta|^2)\,d\zeta \\ &= \widetilde{G}((1,1,1)\cdot\eta, |\eta|^2)\, m_{1,\eta}(\mathbb{R}^3) = \widetilde{G}((1,1,1)\cdot\eta, |\eta|^2). \end{aligned} \tag{2.5}$$

In the second equality we used the fact that the first δ-measure forces $(1,1,1)\cdot\zeta = (1,1,1)\cdot\eta$ and the second δ-measure forces $|\zeta|^2 = |\eta|^2$. For the last equality, we used Lemma 2.2.i.

The equality (2.5) shows that $2\sqrt{3}A$ acts, indeed, as the identity on a dense set of function in the range of the projection operator P_1. Since it also maps into the range of P_1, we conclude that $A = \frac{1}{2\sqrt{3}}P_1$ which, together with (2.2) to (2.4), finishes the proof of part a) of the Representation Theorem 1.3.

2.2. **Two dimensional case.** The proof in the two dimensional case is very similar to the one in one space dimension. In this case $p(d) = 4$. Introduce the notation $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2\times\mathbb{R}^2$, similarly $\zeta = (\zeta_1, \zeta_2)$, and $f\otimes f(\eta) = f(\eta_1)f(\eta_2)$. Squaring (2.1) with $d = 2$, one sees

$$|u(t,x)|^4 = \frac{1}{(4\pi t)^4}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} e^{-i\frac{x\cdot(\eta_1+\eta_2-\zeta_1-\zeta_2)}{2t}} e^{i\frac{|\eta|^2-|\zeta|^2}{4t}} \overline{f\otimes f(\eta)} f\otimes f(\zeta)\,d\zeta d\eta.$$

As before, we integrate this with respect to the now two dimensional variable x and make the substitution $x = 2tz$. Using the fact that for the Dirac measure on $\mathbb{R}^2$, one has $\delta(\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-ix\cdot\xi}\,dx$, and then integrating with respect to t, making the substitution $t = 1/(4\tau)$, we arrive at

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} |u(t,x)|^4\,dxdt = \frac{1}{2\pi}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \delta(\eta_1+\eta_2-\zeta_1-\zeta_2)\delta(|\eta|^2-|\zeta|^2)\overline{f\otimes f(\eta)}f\otimes f(\zeta)\,d\zeta d\eta \tag{2.6}$$

This is nearly exactly the same expression as in the on dimensional case. This time, however, the first δ-function denotes a Dirac measure on $\mathbb{R}^2$. Recall that $\alpha_1 = (1,0,1,0)$ and $\alpha_2 = (0,1,0,1)$. Then $\eta_1+\eta_2-\zeta_1-\zeta_2 = (\alpha_1\cdot(\eta-\zeta), \alpha_2\cdot(\eta-\zeta)) \in \mathbb{R}^2$. So we can rewrite (2.6) as

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} |u(t,x)|^4\,dxdt = \frac{1}{2\pi}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \delta(\alpha_1\cdot(\eta-\zeta))\delta(\alpha_2\cdot(\eta-\zeta))\delta(|\eta|^2-|\zeta|^2)\overline{f\otimes f(\eta)}f\otimes f(\zeta)\,d\zeta d\eta \tag{2.7}$$

For $F, G \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ define the symmetric quadratic form

$$Q_2(F,G) := \frac{1}{2\pi}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \overline{F(\eta)}\delta(\alpha_1\cdot(\eta-\zeta))\delta(\alpha_2\cdot(\eta-\zeta))\delta(|\eta|^2-|\zeta|^2)G(\zeta)\,d\zeta d\eta \tag{2.8}$$

Similarly to the one-dimensional case, $Q_2(F,G) = \langle F, A_2 G\rangle_{L^2(\mathbb{R}^4)}$, where the operator A_2 can be read off from (2.8),

$$(A_2G)(\eta) := \frac{1}{2\pi}\int_{\mathbb{R}^4} \delta(\alpha_1\cdot(\eta-\zeta))\delta(\alpha_2\cdot(\eta-\zeta))\delta(|\eta|^2-|\zeta|^2)G(\zeta)\,d\zeta \tag{2.9}$$

Assume, for the moment,

Lemma 2.3. *The operator A_2 defined in* (2.9) *maps $\mathcal{C}_0^\infty(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$ and extends to a bounded linear operator on all of $L^2(\mathbb{R}^4)$.*

Lemma 2.4. (i) *For all $\eta \in \mathbb{R}^4$, the measure*

$$m_{2,\eta}(d\zeta) = \tfrac{2}{\pi}\delta(\alpha_1\cdot(\eta-\zeta))\delta(\alpha_2\cdot(\eta-\zeta))\delta(|\eta|^2-|\zeta|^2)\,d\zeta$$

is a probability measure on $\mathbb{R}^4$.
(ii) *For all Borel measurable sets $B \subset \mathbb{R}^4$, we have*

$$\int_{\mathbb{R}^3} m_{2,\eta}(B)\,d\eta = |B|$$

where $|B|$ denotes the Lebesgue measure of B.

With this we can argue exactly as in the one-dimensional case. By inspection, $(A_2G)(\eta) = \widetilde{G}(\alpha_1\cdot\eta, \alpha_2\cdot\eta, |\eta|^2)$ for some function $\widetilde{G}$ on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$. Thus using (1.7), we see that again A_2 maps $\mathcal{C}_0^\infty(\mathbb{R}^4)$ into L_2, the range of P_2. Moreover, the set of functions G of the form $G(\zeta) = \widetilde{G}(\alpha_1 \cdot \zeta, \alpha_2 \cdot \zeta, |\zeta|^2)$ for some $\widetilde{G} \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+)$ is dense in the range of P_2 and

$$\begin{aligned}(4A_2G)(\eta) &= \int_{\mathbb{R}^4} \widetilde{G}(\alpha_1 \cdot \zeta, \alpha_2 \cdot \zeta, |\zeta|^2)\, m_{2,\eta}(d\zeta) \\ &= \widetilde{G}(\alpha_1 \cdot \eta, \alpha_2 \cdot \eta, |\eta|^2)\, m_{2,\eta}(\mathbb{R}^4) = G(\eta)\end{aligned} \tag{2.10}$$

So again, the bounded operator A_2 maps $\mathcal{C}_0^\infty(\mathbb{R}^4)$ into the range of P_2 and on the range of P_2 $4A_2$ acts as the identity. Thus $A_2 = \frac{1}{4}P_2$. Together with (2.7) to (2.9), this proves the second part of Theorem 1.3. It remains to do the

Proof of Lemma 2.4. For the first part, write $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ and rotate the ζ coordinates so that $\alpha_1 = (1, 0, 1, 0)$ is in the ζ_1 direction and $\alpha_2 = (0, 1, 0, 1)$ is in the ζ_2 direction. Also, let $\tau_1 = \alpha_1\eta = \eta_1 + \eta_3$, $\tau_2 = \alpha_2\eta = \eta_2 + \eta_4$, and $\xi = |\eta|^2$ and note that

$$\xi - \frac{\tau_1^2}{2} - \frac{\tau_2^2}{2} = \frac{1}{2}((\eta_1 - \eta_2)^2 + (\eta_2 - \eta_4)^2) \geq 0.$$

Then

$$\begin{aligned}m_{2,\eta}(\mathbb{R}^4) &= \frac{2}{\pi} \int_{\mathbb{R}^4} \delta(\tau_1 - \alpha_1 \cdot \zeta)\delta(\tau_2 - \alpha_2 \cdot \zeta)\delta(\xi - |\zeta|^2)\, d\zeta \\ &= \frac{1}{\pi} \int_{\mathbb{R}^4} \delta(\tau_1 - \sqrt{2}\zeta_1)\delta(\tau_2 - \sqrt{2}\zeta_2)\delta(\xi - |\zeta|^2)\, \sqrt{2}d\zeta_1\, \sqrt{2}d\zeta_2\, d(\zeta_3, \zeta_4) \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \delta(\xi - \frac{\tau_1^2}{2} - \frac{\tau_2^2}{2} - |(\zeta_3, \zeta_4)|^2) d(\zeta_3, \zeta_4) \\ &= 2 \int_0^\infty \delta(\frac{1}{2}((\eta_1 - \eta_2)^2 + (\eta_2 - \eta_4)^2) - r^2)\, r\, dr = 1\end{aligned}$$

The proof of the second part follows from this and the Fubini-Tonelli theorem as in the proof of the second part of Lemma 2.2. ∎

Proof of Lemma 2.3. Using $(A_2G)(\eta) = \frac{1}{4} \int_{\mathbb{R}^4} G(\zeta)\, m_{2,\eta}(d\zeta)$ and Lemma 2.4 this is exactly the same calculation as in the proof of Lemma 2.1. ∎

3. All maximizers are Gaussians

In this section we finish the proof of Theorem 1.5. First, we discuss the one-dimensional case. Recall that by the Representation Theorem 1.3, one has equality in the Strichartz inequality for the initial condition $f \in L^2(\mathbb{R})$ if and only if the the function $\mathbb{R}^3 \ni \eta \mapsto h(\eta) = f(\eta_1)f(\eta_2)f(\eta_3)$ is in the range of the orthogonal projection operator P_1, that is, it is invariant under rotations of $\mathbb{R}^3$ which keep the $(1, 1, 1)$ direction fixed. A convenient way to write down a matrix representation for such a rotation is as follows. The three vectors

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \text{ and } v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

obviously form an orthonormal basis of $\mathbb{R}^3$. Any matrix representing a rotation of $\mathbb{R}^3$ keeping the v_1-direction fixed is given by

$$M(\theta) := v_1\otimes v_1+\cos(\theta)v_2\otimes v_2-\sin(\theta)v_2\otimes v_3+\sin(\theta)v_3\otimes v_2+\cos(\theta)v_3\otimes v_3. \quad (3.1)$$

for some $\theta \in [0, 2\pi)$. Here $v_1 \otimes v_2$ etc., is the usual tensor product of two vectors. For example,

$$v_1 \otimes v_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1,1,1) = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note also that $M(0) = v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3$ is the identity matrix on $\mathbb{R}^3$, since v_1, v_2, v_3 form an orthonormal basis of $\mathbb{R}^3$.

If h is a differentiable function on $\mathbb{R}^3$ invariant under rotations of $\mathbb{R}^3$ keeping v_1 fixed, then one must have $h(\eta) = h(M(\theta)\eta)$, where $M(\theta)\eta$ is defined by the usual matrix multiplication. In particular,

$$\frac{\partial}{\partial\theta}h(M(\theta)\eta) = 0 \quad (3.2)$$

for all θ and $\eta \in \mathbb{R}^3$. By the chain rule,

$$\frac{\partial}{\partial\theta}h(M(\theta)\eta) = \nabla h(\eta) \circ \frac{\partial M(\theta)}{\partial\theta}(\eta).$$

Here $\nabla h = (\frac{\partial}{\partial\eta_1}h, \frac{\partial}{\partial\eta_2}h, \frac{\partial}{\partial\eta_3}h)$. Since $M(0)$ is the identity matrix and

$$\frac{\partial M(\theta)}{\partial\theta}\Big|_{\theta=0} = (v_3 \otimes v_2 - v_2 \otimes v_3) = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

any function h obeying (3.2) must have

$$\begin{aligned} 0 = Lh :=& -\sqrt{3}\frac{\partial}{\partial\theta}h(M(\theta)\eta)\Big|_{\theta=0} \\ &= \big((\eta_2-\eta_3)\frac{\partial}{\partial\eta_1} - (\eta_1-\eta_3)\frac{\partial}{\partial\eta_2} + (\eta_1-\eta_2)\frac{\partial}{\partial\eta_3}\big)h. \end{aligned} \quad (3.3)$$

Now we come to the

Proof of Theorem 1.5 in one space dimension. Let $f \in L^2(\mathbb{R})$ be a non trivial maximizer for the Strichartz inequality. By the Representation Theorem, the function $f\otimes f\otimes f$ is in the range of P_1, hence invariant under rotations which keep the v_1 direction fixed.

Step 1: Assume in addition that f is differentiable and never vanishes. Then f is a Gaussian. Indeed, in this case $h(\eta) = f(\eta_1)f(\eta_2)f(\eta_3)$ is a differentiable function invariant under rotations of $\mathbb{R}^3$ keeping v_1 fixed, so (3.3) holds for h. Since, by assumption, f never vanishes, one can further simplify (3.3) to

$$0 = (\eta_2-\eta_3)g'(\eta_1) - (\eta_1-\eta_3)g'(\eta_2) + (\eta_1-\eta_2)g'(\eta_3) \quad (3.4)$$

where we also introduced $g'(x) = f'(x)/f(x)$, the logarithmic derivative of f. Differentiating (3.4) with respect to η_1, say, gives

$$g''(\eta_1) = \frac{g'(\eta_2) - g'(\eta_3)}{\eta_2 - \eta_3} \quad \text{for } \eta_2 \neq \eta_3. \tag{3.5}$$

Since η_1, η_2, η_3 are independent variables, g'' must be constant, so g is a second order polynomial. Hence f must be a Gaussian. This settles the one-dimensional case, as soon as one knows that the maximizer is smooth and non-vanishing.

To proceed with the proof in the general case we follow an argument of Carlen [1]. We need a bit more notation: Denote by P_t convolution on $\mathbb{R}^3$ with the kernel $(2\pi t)^{-3/2} \exp(-|\eta|^2/(2t))$, and by Q_t convolution on $\mathbb{R}$ with the kernel $(2\pi t)^{-1/2} \exp(-|x|^2/(2t))$.

Step 2: Let $f \in L^2(\mathbb{R})$ with $h = f \otimes f \otimes f$ in the range of P_1. Assume that $Q_t(f)$ never vanishes. Then f is a Gaussian.

Indeed, by rotational invariance of P_t, the function $P_t h$ inherits the rotational invariance of h. Moreover, by the product structure of the kernel of P_t,

$$P_t h = (Q_t f) \otimes (Q_t f) \otimes (Q_t f). \tag{3.6}$$

$Q_t f$ is certainly a smooth function for $t > 0$. Since by assumption $Q_t f$ never vanishes, we can use Step 1 to conclude that $Q_t f$ is a Gaussian. Hence $f = \lim_{t\to 0} Q_t f$ is a limit of Gaussians and hence a Gaussian. To finish the proof, we note

Step 3: Assume that f is a maximizer in the Strichartz inequality. Then $Q_t f$ never vanishes for all $t > 0$.

Indeed, take the modulus in (3.6) and apply P_s again. This gives

$$P_s|P_t h| = Q_s|Q_t f| \otimes Q_s|Q_t f| \otimes Q_s|Q_t f|$$

Again, $P_s|h_t|$ inherits all rotational symmetries of $P_t h$, in particular, those of h. Since $Q_t f \to f$ in $L^2(\mathbb{R}^3)$ as $t \to 0$, we see that $Q_t f$ is not the zero function for small t. Thus, since convolution with Gaussians improves positivity, $Q_s|Q_t f|$ is a strictly positive smooth function. By Step 2, we can conclude that $|Q_t f|$ is a Gaussian. Hence $Q_t f$ never vanishes for small t.

This finishes the proof of Theorem 1.5 in the one-dimensional case. ∎

For the two dimensional case, recall that the Representation Theorem (1.3) says that if a non-trivial function $f \in L^2(\mathbb{R}^2)$ is a maximizer of the Strichartz inequality in two dimensions then the map $\mathbb{R}^4 \ni \eta \mapsto h(\eta) = f(\eta_1, \eta_2) f(\eta_3, \eta_4)$ is is in the range of P_2.

As a preliminary result let us note that this immediately implies that f is a product of two one-dimensional functions f_1 and f_2. Indeed, since h defined above is in the range of P_2, it is of the form $h(\eta) = \tilde{h}(\alpha_1 \cdot \eta, \alpha_2 \cdot \eta, |\eta|^2)$ which is invariant under exchanging the η_1 and η_3 coordinates. This can be checked directly, or by introducing the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since α_1 and α_1 are eigenvectors of M with eigenvalue one and $|M\eta|^2 = |\eta|^2$, the invariance $h(\eta) = h(M\eta)$ holds for any $h \in \mathrm{Ran}(P_2)$. This gives

$$f(\eta_1, \eta_2) f(\eta_3, \eta_4) = f(\eta_3, \eta_2) f(\eta_1, \eta_4)$$

for almost all η. Since $f \in L^2(\mathbb{R}^2)$, it is locally integrable. Pick a square $Q = I_1 \times I_2 \subset \mathbb{R}^2$ such that $J = \int_Q f(\eta_1, \eta_2)\, d\eta_1 d\eta_2 \neq 0$. Then integrating the above equality with respect to η_1 and η_2 over Q and using Fubini's theorem shows

$$J f(\eta_3, \eta_4) = \int_{I_2} f(\eta_3, \eta_2)\, d\eta_2 \int_{I_1} f(\eta_1, \eta_4)\, d\eta_1.$$

So every maximizer f in the two dimensional Strichartz inequality factorizes into a product of two one-dimensional functions, $f = f_1 \otimes f_2$ with $f_j \in L^2(\mathbb{R})$.

Now assume that $\mathbb{R}^4 \ni \eta \to h(\eta)$ is any differentiable function invariant under rotations of $\mathbb{R}^4$ which keep the $\alpha_1 = (1, 0, 1, 0)$ and $\alpha_2 = (0, 1, 0, 1)$ directions fixed. A convenient parametrization of these rotations is as follows. The four vectors

$$\alpha_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \alpha_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \text{ and } \alpha_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

form an orthonormal basis of $\mathbb{R}^4$. Any rotation keeping the α_1 and α_2 directions fixed is given by the matrix

$$\begin{aligned} M(\theta) := {} & \alpha_1 \otimes \alpha_1 + \alpha_2 \otimes \alpha_2 + \cos(\theta) \alpha_3 \otimes \alpha_3 - \sin(\theta) \alpha_3 \otimes \alpha_4 \\ & + \sin(\theta) \alpha_4 \otimes \alpha_3 + \cos(\theta) \alpha_4 \otimes \alpha_4 \end{aligned}$$

for some $\theta \in [0, 2\pi)$.

As in the three dimensional case, one concludes that h must satisfy the differential equation

$$\begin{aligned} 0 = Lh(\eta) := {} & -2 \frac{\partial h(M(\theta))}{\partial \theta} \Big|_{\theta=0} \\ = {} & \big[(\eta_2 - \eta_4)\partial_1 - (\eta_1 - \eta_3)\partial_2 - (\eta_2 - \eta_4)\partial_3 + (\eta_1 - \eta_3)\partial_4 \big] h(\eta) \end{aligned} \tag{3.7}$$

We can now give the

Proof of Theorem 1.5 in two space dimensions. Let f be a non trivial maximizer in the two dimensional Strichartz inequality. Then by the above discussion $f(\eta_1, \eta_2) = f_1(\eta_1) f_2(\eta_2)$. Moreover, the function

$$h(\eta) = f_1(\eta_1) f_2(\eta_2) f_1(\eta_3) f_2(\eta_4). \tag{3.8}$$

is in the range of P_2, by the Representation Theorem, in particular, invariant under the rotations $M(\theta)$ defined above.

Now assume, in addition, that f_j, $j = 1, 2$, is smooth and never vanishes. Then h obeys the differential equation (3.7). Introducing the logarithmic derivatives $g_j' = f_j'/f_j$, we can rewrite this as

$$0 = (\eta_2 - \eta_4)g_1'(\eta_1) - (\eta_1 - \eta_3)g_2'(\eta_2) - (\eta_2 - \eta_4)g_1'(\eta_3) + (\eta_1 - \eta_3)g_2'(\eta_4). \quad (3.9)$$

Differentiating with respect to η_1 yields

$$0 = (\eta_2 - \eta_4)g_1''(\eta_1) - g_2'(\eta_2) + g_2'(\eta_4). \quad (3.10)$$

and differentiating this again with respect to η_2 gives

$$g_1''(\eta_1) = g_2''(\eta_2). \quad (3.11)$$

Thus, since η_1 and η_2 are independent variables, both second logarithmic derivatives are constant and equal. So f is the product of two one-dimensional Gaussians with the same covariance. This proves the two-dimensional version of Theorem 1.5 if f is smooth and non-vanishing. The general case follows from this by an argument very similar to the one used in the proof of the one-dimensional version, one has to use convolution with a rotationally symmetric Gaussian on $\mathbb{R}^4$ and use its invariance under rotations of $\mathbb{R}^4$ and its product structure as in the previous proof. ∎

4. Extremizers for $d \geq 3$

Unfortunately neither Foschi's method nor ours does directly apply to dimensions three and higher. There is, however, a natural procedure to verify if there exist Gaussians *extremizers* in higher dimension. Assume first that the Gaussian

$$u(x) = \alpha e^{\frac{-|x|^2}{2\beta}}$$

is an extremizer, then it must solve the corresponding Euler-Lagrange equation for (1.4),

$$\lambda u = \int_{\mathbb{R}} e^{-is\Delta}(|e^{is\Delta}u|^{\frac{4}{d}} e^{is\Delta}u)ds.$$

We observe that by using invariance under Schrödinger evolution, we can adjust β to be real. Secondly by using dilation invariance

$$u(x) \to \mu^{d/2}u(\mu x),$$

we may adjust $\beta = 1$. Finally, by fixing an appropriate L^2 norm, we let $\alpha = 1$. Therefore, it is sufficient to verify that $u = e^{-|x|^2/2}$ solves the above equation. After direct calculations, we obtain that the Euler-Lagrange equation takes the form

$$\lambda e^{-|x|^2/2} = \int_{\mathbb{R}} \frac{ds}{1+4s^2}\left(\frac{1-i2s}{1-i2qs}\right)^{d/2} \exp\left(-\frac{|x|^2}{2}\frac{q-i2s}{1-i2qs}\right),$$

where $q = 1 + 4/d$. Consider this integral in complex plane with the contour being: $[-R, R] \in \mathbb{R}$ and the semicircle with radius R. It is easy to check that the integral over upper semi-circle vanishes as $R \to \infty$. The contour has a single pole in upper semi-plane $s = i/2$. Therefore, the equation takes the form

$$\lambda e^{-|x|^2/2} = 2\pi i \operatorname{Res}_{s=i/2}(f) \ = 2\pi i \frac{1}{2i}\frac{1}{2}\left(\frac{2}{1+q}\right)^{d/2} e^{-\frac{|x|^2}{2}\frac{q+1}{q+1}}.$$

Choosing

$$\lambda = \frac{\pi}{2}\left(\frac{2}{1+q}\right)^{d/2} = \frac{\pi}{2}\left(\frac{d}{2+d}\right)^{d/2}$$

as the Lagrange multiplier, we obtain the equality. Hence in all dimensions, Gaussians with a rotationally symmetric covariance are solutions of the Euler-Lagrange equation for extremizers in the Strichartz inequality.

Remarks 4.1. (i) The above calculation breaks down if we assume that the maximizer is a Gaussian with a non-rotationally symmetric covariance.

(ii) We cannot rule out the existence on non-Gaussian extremizers for the Strichartz inequality in dimension three and higher. We believe that they do not exist and that all maximizers must be Gaussians. This is in spirit with Lieb's result [7] that Gaussian kernels have only Gaussian extremizers.

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