Decay estimates for Dispersion Management Solitons

Dirk Hundertmark

University of Birmingham and

University of Illinois at Urbana-Champaign

Plan

- The Dispersion Management Functional.

Plan

- The Dispersion Management Functional.
- Physical relevance.

Plan

- The Dispersion Management Functional.
- Physical relevance.
- Decay estimates for Dispersion Managed Solitons:

Plan

- The Dispersion Management Functional.
- Physical relevance.
- Decay estimates for Dispersion Managed Solitons:
  - harmonic analysis meets real space.

Plan

- The Dispersion Management Functional.
- Physical relevance.
- Decay estimates for Dispersion Managed Solitons:
  - harmonic analysis meets real space.
  - Sub-harmonicity rules!

The Dispersion Management functional

Interested in minimizers (for fixed $\|f\|_{L^2(\mathbb{R})}$) of

$$H(f) := \frac{\overline{d}}{2} \int_{\mathbb{R}} |f'|^2 \, dx - \frac{1}{4} \mathcal{Q}(f, f, f, f)$$

The Dispersion Management functional

Interested in minimizers (for fixed $\|f\|_{L^2(\mathbb{R})}$) of

$$H(f) := \frac{\overline{d}}{2}\int_{\mathbb{R}} |f'|^2\,dx - \frac{1}{4}\mathcal{Q}(f,f,f,f)$$

with

$$\begin{aligned}\mathcal{Q}(f,f,f,f) &= \int_0^1 dt \int_{\mathbb{R}} dx |T_t f(x)|^4 \\ &= \int_0^1 \int_{\mathbb{R}} \overline{T_t f(x)} T_t f(x) \overline{T_t f(x)} T_t f(x)\,dxdt\end{aligned}$$

The Dispersion Management functional

Interested in minimizers (for fixed $\|f\|_{L^2(\mathbb{R})}$) of

$$H(f) := \frac{\overline{d}}{2} \int_{\mathbb{R}} |f'|^2 \, dx - \frac{1}{4} \mathcal{Q}(f, f, f, f)$$

with

$$\begin{aligned} \mathcal{Q}(f, f, f, f) &= \int_0^1 dt \int_{\mathbb{R}} dx |T_t f(x)|^4 \\ &= \int_0^1 \int_{\mathbb{R}} \overline{T_t f(x)} T_t f(x) \overline{T_t f(x)} T_t f(x) \, dx dt \end{aligned}$$

and

$$T_t = e^{it\partial_x^2}.$$

Physical relevance

Physical relevance

The equation describing the amplitude u of a signal in a glass-fibre cable is given by a cubic non-linear Schrödinger equation

Physical relevance

The equation describing the amplitude u of a signal in a glass-fibre cable is given by a cubic non-linear Schrödinger equation

$$i\partial_t v = -d(t)\partial_x^2 v - c|u|^2 v$$

Physical relevance

The equation describing the amplitude u of a signal in a glass-fibre cable is given by a cubic non-linear Schrödinger equation

$$i\partial_t v = -d(t)\partial_x^2 v - c|u|^2 v$$

Here $d = d(t)$ is the varying dispersion along the fibre.

Physical relevance

The equation describing the amplitude u of a signal in a glass-fibre cable is given by a cubic non-linear Schrödinger equation

$$i\partial_t v = -d(t)\partial_x^2 v - c|u|^2 v$$

Here $d = d(t)$ is the varying dispersion along the fibre.

Problem: Low power pulses disperse along the fibre. High power pulses (solitons) interact strongly due to the non-linearity.

Physical relevance

The equation describing the amplitude u of a signal in a glass-fibre cable is given by a cubic non-linear Schrödinger equation

$$i\partial_t v = -d(t)\partial_x^2 v - c|u|^2 v$$

Here $d = d(t)$ is the varying dispersion along the fibre.

Problem: Low power pulses disperse along the fibre. High power pulses (solitons) interact strongly due to the non-linearity.

Dispersion Management: Stay in the linear regime, where pulses do not interact with each other.

Physical relevance

The equation describing the amplitude u of a signal in a glass-fibre cable is given by a cubic non-linear Schrödinger equation

$$i\partial_t v = -d(t)\partial_x^2 v - c|u|^2 v$$

Here $d = d(t)$ is the varying dispersion along the fibre.

Problem: Low power pulses disperse along the fibre. High power pulses (solitons) interact strongly due to the non-linearity.

Dispersion Management: Stay in the linear regime, where pulses do not interact with each other.

Stabilize pulses by strongly periodically varying the dispersion d.

Physical relevance

Effective equation for the average amplitude v

$$i\partial_t v = -\overline{d}\partial_x^2 v - Q(v,v,v) \qquad \text{Gabitov-Turytsin (1996)}$$

Physical relevance

Effective equation for the average amplitude v

$$i\partial_t v = -\overline{d}\partial_x^2 v - Q(v,v,v) \qquad \text{Gabitov-Turytsin (1996)}$$

with $\overline{d} = \text{average}(d)$

$$\text{and} \qquad Q(u,u,u) = \int_0^1 T_t^{-1}\left[(\overline{T_t v} T_t v) T_t v\right] dt$$

Physical relevance

Effective equation for the average amplitude v

$$i\partial_t v = -\overline{d}\partial_x^2 v - Q(v, v, v) \qquad \text{Gabitov-Turytsin (1996)}$$

with $\overline{d} =$ average(d)

$$\text{and} \qquad Q(u, u, u) = \int_0^1 T_t^{-1}\left[(\overline{T_t v} T_t v) T_t v\right] dt$$

Separation of variables: $v(t, x) = e^{i\omega t} f(x)$ yields

$$-\omega f = -\alpha f'' - Q(f, f, f)$$

Physical relevance

Solution of this equation are critical points of

$$H(f) = \frac{\overline{d}}{2} \int_{\mathbb{R}} |f'(x)|^2 \, dx - \frac{1}{4} \mathcal{Q}(f, f, f, f)$$

for fixed $\|f\|_{L^2(\mathbb{R})} = \lambda$ (power of the pulse).

Physical relevance

Solution of this equation are critical points of

$$H(f) = \frac{\overline{d}}{2} \int_{\mathbb{R}} |f'(x)|^2 \, dx - \frac{1}{4} \mathcal{Q}(f, f, f, f)$$

for fixed $\|f\|_{L^2(\mathbb{R})} = \lambda$ (power of the pulse).

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) \, dx dt$$

Physical relevance

Solution of this equation are critical points of

$$H(f) = \frac{\overline{d}}{2} \int_{\mathbb{R}} |f'(x)|^2 \, dx - \frac{1}{4} \mathcal{Q}(f, f, f, f)$$

for fixed $\|f\|_{L^2(\mathbb{R})} = \lambda$ (power of the pulse).

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) \, dx dt$$

This idea turned out to be enormously successful, creating optical waveguides with record breaking bit-rates (Tera bits$/$s over a 10000 kilometer glass-fibre cable!).

Physical relevance

Solution of this equation are critical points of

$$H(f) = \frac{\overline{d}}{2} \int_{\mathbb{R}} |f'(x)|^2 \, dx - \frac{1}{4} \mathcal{Q}(f,f,f,f)$$

for fixed $\|f\|_{L^2(\mathbb{R})} = \lambda$ (power of the pulse).

$$\mathcal{Q}(f_1,f_2,f_3,f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) \, dx dt$$

This idea turned out to be enormously successful, creating optical waveguides with record breaking bit-rates (Tera bits$/$s over a 10000 kilometer glass-fibre cable!).

Now widely used commercially: e.g., fast internet.

Rigorous results

Rigorous results

- V. Zharnitsky, E. Grenier, C.K.R.T. Jones, S. K. Turitsyn (2001): minimizers exist in $H^1(\mathbb{R})$ for positive average dispersion $\overline{d} > 0$.

Rigorous results

- V. Zharnitsky, E. Grenier, C.K.R.T. Jones, S. K. Turitsyn (2001): minimizers exist in $H^1(\mathbb{R})$ for positive average dispersion $\overline{d} > 0$.
- Using $Q : (H^s)^3 \rightarrow H^s$ and $(-\omega + \overline{d}\partial_x^2)^{-1} : H^s \rightarrow H^{s+2}$, one can bootstrap the DM equation: every solution is automatically smooth.

Rigorous results

- V. Zharnitsky, E. Grenier, C.K.R.T. Jones, S. K. Turitsyn (2001): minimizers exist in $H^1(\mathbb{R})$ for positive average dispersion $\overline{d} > 0$.
- Using $Q : (H^s)^3 \to H^s$ and $(-\omega + \overline{d}\partial_x^2)^{-1} : H^s \to H^{s+2}$, one can bootstrap the DM equation: every solution is automatically smooth.
- Loss of compactness in the critical case $\overline{d} = 0$ (zero average dispersion)

Rigorous results

- V. Zharnitsky, E. Grenier, C.K.R.T. Jones, S. K. Turitsyn (2001): minimizers exist in $H^1(\mathbb{R})$ for positive average dispersion $\overline{d} > 0$.
- Using $Q : (H^s)^3 \to H^s$ and $(-\omega + \overline{d}\partial_x^2)^{-1} : H^s \to H^{s+2}$, one can bootstrap the DM equation: every solution is automatically smooth.
- Loss of compactness in the critical case $\overline{d} = 0$ (zero average dispersion)
  - Kunze (2004): Existence of a minimizer for $\overline{d} = 0$

Rigorous results

- V. Zharnitsky, E. Grenier, C.K.R.T. Jones, S. K. Turitsyn (2001): minimizers exist in $H^1(\mathbb{R})$ for positive average dispersion $\overline{d} > 0$.
- Using $Q : (H^s)^3 \to H^s$ and $(-\omega + \overline{d}\partial_x^2)^{-1} : H^s \to H^{s+2}$, one can bootstrap the DM equation: every solution is automatically smooth.
- Loss of compactness in the critical case $\overline{d} = 0$ (zero average dispersion)
  - Kunze (2004): Existence of a minimizer for $\overline{d} = 0$
  - Stanislavova 2004: Kunzes minimizer is smooth

Rigorous results

- V. Zharnitsky, E. Grenier, C.K.R.T. Jones, S. K. Turitsyn (2001): minimizers exist in $H^1(\mathbb{R})$ for positive average dispersion $\overline{d} > 0$.
- Using $Q : (H^s)^3 \to H^s$ and $(-\omega + \overline{d}\partial_x^2)^{-1} : H^s \to H^{s+2}$, one can bootstrap the DM equation: every solution is automatically smooth.
- Loss of compactness in the critical case $\overline{d} = 0$ (zero average dispersion)
  - Kunze (2004): Existence of a minimizer for $\overline{d} = 0$
  - Stanislavova 2004: Kunzes minimizer is smooth
- **Conjecture:** No solution exists for negative average dispersion.

Boundedness

Even for $\overline{d} = 0$, the functional $H(f)$ is bounded below:

$$H(f) = -\mathcal{Q}(f, f, f, f) \geq -\text{const}\,\|f\|^4$$

Boundedness

Even for $\overline{d} = 0$, the functional $H(f)$ is bounded below:

$$H(f) = -\mathcal{Q}(f,f,f,f) \geq -\text{const}\,\|f\|^4$$

$$\mathcal{Q}(f,f,f,f) = \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^4 \, dxdt = \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^{3+1} \, dxdt$$

Boundedness

Even for $\overline{d} = 0$, the functional $H(f)$ is bounded below:

$$H(f) = -\mathcal{Q}(f,f,f,f) \geq -\text{const}\,\|f\|^4$$

$$\mathcal{Q}(f,f,f,f) = \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^4 \, dx dt = \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^{3+1} \, dx dt$$

now use

$$a^3 b \leq \frac{1}{2}\big(\varepsilon^{-1} a^6 + \varepsilon b^2\big) \quad \text{Cauchy-Schwarz with epsilon}$$

Boundedness

Even for $\overline{d} = 0$, the functional $H(f)$ is bounded below:

$$H(f) = -\mathcal{Q}(f,f,f,f) \geq -\text{const}\,\|f\|^4$$

$$\begin{aligned}\mathcal{Q}(f,f,f,f) &= \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^4 \, dx dt = \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^{3+1} \, dx dt \\ &\leq \frac{1}{2}\Big(\varepsilon^{-1} \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^6 \, dx dt + \varepsilon \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^2 \, dx dt\Big)\end{aligned}$$

Boundedness

Even for $\overline{d} = 0$, the functional $H(f)$ is bounded below:

$$H(f) = -\mathcal{Q}(f,f,f,f) \geq -\text{const}\,\|f\|^4$$

$$\begin{aligned}
\mathcal{Q}(f,f,f,f) &= \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^4 \, dx dt = \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^{3+1} \, dx dt \\
&\leq \frac{1}{2}\Big(\varepsilon^{-1} \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^6 \, dx dt + \varepsilon \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^2 \, dx dt\Big) \\
&\leq \frac{1}{2}\Big(\varepsilon^{-1} S_1^6 \|f\|^6 + \varepsilon \|f\|^2\Big)
\end{aligned}$$

using one-dim Strichartz and unicity of T_t.

Boundedness

Even for $\overline{d} = 0$, the functional $H(f)$ is bounded below:

$$H(f) = -\mathcal{Q}(f,f,f,f) \geq -\text{const}\,\|f\|^4$$

$$\begin{aligned}
\mathcal{Q}(f,f,f,f) &= \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^4 \, dx dt = \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^{3+1} \, dx dt \\
&\leq \frac{1}{2}\Big(\varepsilon^{-1} \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^6 \, dx dt + \varepsilon \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^2 \, dx dt \Big) \\
&\leq \frac{1}{2}\Big(\varepsilon^{-1} S_1^6 \|f\|^6 + \varepsilon \|f\|^2 \Big) \\
&= S_1^3 \|f\|^4 \quad \text{with the choice } \varepsilon = S_1^3 \|f\|^2.
\end{aligned}$$

Decay conjecture

Decay conjecture

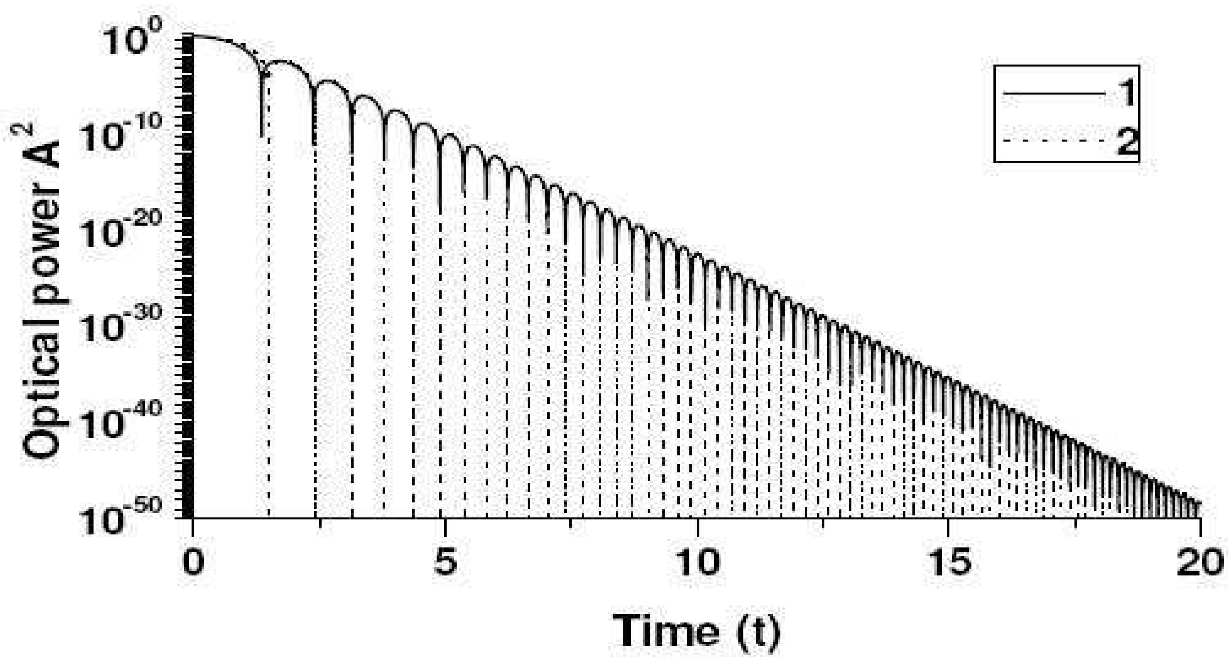
P. Lushnikov (2004) gave convincing arguments that the dispersion managed soliton should look like

$$f(x) \sim \cos(x^2(a_0 + O(x^{-1}))) \exp(-b|x|) \quad \text{as } x \to \infty.$$

Decay conjecture

P. Lushnikov (2004) gave convincing arguments that the dispersion managed soliton should look like

$$f(x) \sim \cos(x^2(a_0 + O(x^{-1})))\exp(-b|x|) \quad \text{as } x \to \infty.$$



Regularity

Theorem (100DM and Young-Ran Lee (2007)). *Any weak solution of*

$$f = Q(f, f, f)$$

is a Schwartz function. That is,

$$\sup_x (1 + |x|)^m |D^n f(x)| < \infty$$

for all $n, m \in \mathbb{N}_0$.

Regularity

Theorem (100DM and Young-Ran Lee (2007)). *Any weak solution of*

$$f = Q(f, f, f)$$

is a Schwartz function. That is,

$$\sup_x (1 + |x|)^m |D^n f(x)| < \infty$$

for all $n, m \in \mathbb{N}_0$*.*

Recall: f is a weak solution if

$$\langle g, f\rangle = \langle g, Q(f, f, f)\rangle = \mathcal{Q}(g, f, f, f)$$

for all $g \in L^2(\mathbb{R})$.

Estimates on the tail

Let $f \in L^2(\mathbb{R})$ be a weak solution of $f = Q(f, f, f)$. Set

$$\alpha(s) := \Big(\int_{|x| \geq s} |f(x)|^2 \, dx \Big)^{1/2}$$

$$\widehat{\alpha}(s) := \Big(\int_{|x| \geq s} |\widehat{f}(x)|^2 \, dx \Big)^{1/2}$$

Estimates on the tail

Let $f \in L^2(\mathbb{R})$ be a weak solution of $f = Q(f, f, f)$. Set

$$\alpha(s) := \Big(\int_{|x|\geq s} |f(x)|^2 \, dx \Big)^{1/2}$$

$$\widehat{\alpha}(s) := \Big(\int_{|x|\geq s} |\widehat{f}(x)|^2 \, dx \Big)^{1/2}$$

Note: α and $\widehat{\alpha}$ are decreasing and tend to zero at infinity.

Estimates on the tail

Let $f \in L^2(\mathbb{R})$ be a weak solution of $f = Q(f,f,f)$. Set

$$\alpha(s) := \Big(\int_{|x|\geq s} |f(x)|^2 \, dx \Big)^{1/2}$$

$$\widehat{\alpha}(s) := \Big(\int_{|x|\geq s} |\widehat{f}(x)|^2 \, dx \Big)^{1/2}$$

Note: α and $\widehat{\alpha}$ are decreasing and tend to zero at infinity.

Theorem.

$$\alpha(s) \lesssim (\alpha(s/3))^3 + \frac{1}{\sqrt{s}} \alpha(0)^2 \alpha(s/3)$$

$$\widehat{\alpha}(s) \lesssim (\widehat{\alpha}(s/3))^3 + \frac{1}{\sqrt{s}} \widehat{\alpha}(0)^2 \widehat{\alpha}(s/3)$$

How to use this

How to use this

If $\alpha \to 0$ at infinity and $\alpha(s) \lesssim (\alpha(s))^3$. Then α has compact support!

How to use this

Since

$$\alpha(s) \lesssim (\alpha(s/3))^3 + \frac{1}{\sqrt{s}}\alpha(s/3)$$

How to use this

Since

$$\alpha(s) \lesssim (\alpha(s/3))^3 + \frac{1}{\sqrt{s}}\alpha(s/3)$$

one gets

$$\chi_{[d,\infty)}(s/3)\alpha(s) \lesssim (\alpha(s/3))^2\chi_{[d,\infty)}(s/3)\alpha(s/3)+\frac{1}{\sqrt{s}}\chi_{[d,\infty)}(s/3)\alpha(s/3)$$

How to use this

Since

$$\alpha(s) \lesssim (\alpha(s/3))^3 + \frac{1}{\sqrt{s}}\alpha(s/3)$$

one gets

$$\chi_{[d,\infty)}(s/3)\alpha(s) \lesssim (\alpha(s/3))^2\chi_{[d,\infty)}(s/3)\alpha(s/3)+\frac{1}{\sqrt{s}}\chi_{[d,\infty)}(s/3)\alpha(s/3)$$

Note $\chi_{[d,\infty)}(s/3) = \chi_{[d,\infty)}(s) - \chi_{[d,3d)}(s)$.

How to use this

Since

$$\alpha(s) \lesssim (\alpha(s/3))^3 + \frac{1}{\sqrt{s}}\alpha(s/3)$$

one gets

$$\chi_{[d,\infty)}(s/3)\alpha(s) \lesssim (\alpha(s/3))^2\chi_{[d,\infty)}(s/3)\alpha(s/3)+\frac{1}{\sqrt{s}}\chi_{[d,\infty)}(s/3)\alpha(s/3)$$

Note $\chi_{[d,\infty)}(s/3) = \chi_{[d,\infty)}(s) - \chi_{[d,3d)}(s)$. Put $\alpha_d := \chi_{[d,\infty)}\alpha$.

How to use this

Since

$$\alpha(s) \lesssim (\alpha(s/3))^3 + \frac{1}{\sqrt{s}}\alpha(s/3)$$

one gets

$$\chi_{[d,\infty)}(s/3)\alpha(s) \lesssim (\alpha(s/3))^2\chi_{[d,\infty)}(s/3)\alpha(s/3)+\frac{1}{\sqrt{s}}\chi_{[d,\infty)}(s/3)\alpha(s/3)$$

Note $\chi_{[d,\infty)}(s/3) = \chi_{[d,\infty)}(s) - \chi_{[d,3d)}(s)$. Put $\alpha_d := \chi_{[d,\infty)}\alpha$. Then

$$\alpha_d(s) \leq C(\alpha_d(s/3))^2\alpha_d(s/3) + C\frac{\alpha_d(s/3)}{\sqrt{s}} + \chi_{[d,\infty)}(s)\alpha(s)$$

How to use this

Hence

$$s^{\gamma}\alpha_d(s) \le C3^{\gamma}(\alpha_d(s/3))^2(s/3)^{\gamma}\alpha_d(s/3) + C\frac{3^{\gamma}(s/3)^{\gamma}\alpha_d(s/3)}{\sqrt{s}}$$

$$+ (3d)^{\gamma}\chi_{[d,3d)}(s)\alpha(s)$$

How to use this

Hence

$$s^\gamma \alpha_d(s) \le C3^\gamma(\alpha_d(s/3))^2 (s/3)^\gamma \alpha_d(s/3) + C\frac{3^\gamma (s/3)^\gamma \alpha_d(s/3)}{\sqrt{s}}$$

$$+ (3d)^\gamma \chi_{[d,3d)}(s)\alpha(s)$$

and

$$\|s^\gamma \alpha_d\|_\infty \le C3^\gamma(\alpha(d))^2 \|s^\gamma \alpha_d\|_\infty + C\frac{3^\gamma \|s^\gamma \alpha_d\|_\infty}{\sqrt{s}}$$

$$+ (3d)^\gamma \alpha(0)$$

How to use this

Rearrange to

$$\Big(1 - C3^{\gamma}\alpha(d)^2 - \frac{C3^{\gamma}}{\sqrt{s}}\Big)\|s^{\gamma}\alpha_d\|_{\infty} \leq (3d)^{\gamma}\alpha(0)$$

How to use this

Rearrange to

$$\left(1 - C3^\gamma \alpha(d)^2 - \frac{C3^\gamma}{\sqrt{s}}\right)\|s^\gamma \alpha_d\|_\infty \le (3d)^\gamma \alpha(0)$$

This gives a finite upper bound on $\|s^\gamma \alpha_d\|_\infty$ as long as d and s are chose so large that

$$C3^\gamma \alpha(d)^2 + \frac{C3^\gamma}{\sqrt{s}} \le \frac{1}{2}.$$

How to use this

Rearrange to

$$\Big(1 - C3^\gamma \alpha(d)^2 - \frac{C3^\gamma}{\sqrt{s}}\Big)\|s^\gamma \alpha_d\|_\infty \leq (3d)^\gamma \alpha(0)$$

This gives a finite upper bound on $\|s^\gamma \alpha_d\|_\infty$ as long as d and s are chose so large that

$$C3^\gamma \alpha(d)^2 + \frac{C3^\gamma}{\sqrt{s}} \leq \frac{1}{2}.$$

In particular, $s \mapsto s^\gamma \alpha(s)$ is bounded or

How to use this

Rearrange to

$$\Big(1 - C3^\gamma \alpha(d)^2 - \frac{C3^\gamma}{\sqrt{s}}\Big) \|s^\gamma \alpha_d\|_\infty \le (3d)^\gamma \alpha(0)$$

This gives a finite upper bound on $\|s^\gamma \alpha_d\|_\infty$ as long as d and s are chose so large that

$$C3^\gamma \alpha(d)^2 + \frac{C3^\gamma}{\sqrt{s}} \le \frac{1}{2}.$$

In particular, $s \mapsto s^\gamma \alpha(s)$ is bounded or

$$\alpha(s) \lesssim s^{-\gamma} \quad \text{for all } \gamma > 0.$$

Corollary. *If* $f \in L^2(\mathbb{R})$ *solves* $f = Q(f, f, f)$ *weakly,*

i) $(1 + x^2)^{\gamma/2} f(x) \in L^2(\mathbb{R})$ *and*

ii) $(1 + k^2)^{\gamma/2} \widehat{f}(k) \in L^2(\mathbb{R})$ *for all* $\gamma > 0$*.*

Corollary. *If* $f \in L^2(\mathbb{R})$ *solves* $f = Q(f,f,f)$ *weakly,*

i) $(1+x^2)^{\gamma/2} f(x) \in L^2(\mathbb{R})$ *and*

ii) $(1+k^2)^{\gamma/2} \widehat{f}(k) \in L^2(\mathbb{R})$ *for all* $\gamma > 0$*.*

Corollary. *Any weak solution* $f \in L^2(\mathbb{R})$ *of* $f = Q(f,f,f)$ *is a Schwartz function.*

Corollary. *If* $f \in L^2(\mathbb{R})$ *solves* $f = Q(f, f, f)$ *weakly,*
i) $(1 + x^2)^{\gamma/2} f(x) \in L^2(\mathbb{R})$ *and*
ii) $(1 + k^2)^{\gamma/2} \widehat{f}(k) \in L^2(\mathbb{R})$ *for all* $\gamma > 0$.

Corollary. *Any weak solution* $f \in L^2(\mathbb{R})$ *of* $f = Q(f, f, f)$ *is a Schwartz function.*

Homework: If $(1 + x^2)^{n/2} f(x) \in L^2(\mathbb{R})$ and $D^n f \in L^2(\mathbb{R})$ for all $n \in \mathbb{N}_0$, then $(1 + x^2)^{n/2} D^m f \in L^2(\mathbb{R})$ for all $n, m \in \mathbb{N}_0$, i.e., f is a Schwartz function.

Corollary. *If* $f \in L^2(\mathbb{R})$ *solves* $f = Q(f, f, f)$ *weakly,*
i) $(1 + x^2)^{\gamma/2} f(x) \in L^2(\mathbb{R})$ *and*
ii) $(1 + k^2)^{\gamma/2} \widehat{f}(k) \in L^2(\mathbb{R})$ *for all* $\gamma > 0$*.*

Corollary. *Any weak solution* $f \in L^2(\mathbb{R})$ *of* $f = Q(f, f, f)$ *is a Schwartz function.*

Homework: If $(1 + x^2)^{n/2} f(x) \in L^2(\mathbb{R})$ and $D^n f \in L^2(\mathbb{R})$ for all $n \in \mathbb{N}_0$, then $(1 + x^2)^{n/2} D^m f \in L^2(\mathbb{R})$ for all $n, m \in \mathbb{N}_0$, i.e., f is a Schwartz function.

Remaining Question: How to get the a-priori bound on the tail distribution?

Recall: $f = Q(f, f, f)$ weakly, i.e,

Recall: $f = Q(f,f,f)$ weakly, i.e,

$$\langle g, f\rangle = \langle g, Q(f,f,f)\rangle = \mathcal{Q}(g,f,f,f)$$

for all $g \in L^2(\mathbb{R})$.

Recall: $f = Q(f, f, f)$ weakly, i.e,

$$\langle g, f \rangle = \langle g, Q(f, f, f) \rangle = \mathcal{Q}(g, f, f, f)$$

for all $g \in L^2(\mathbb{R})$.

Note

$$\alpha(s) = \sup \left\{ |\langle g, f \rangle| : \|g\| = 1, \text{supp}(g) \subset \{|x| \geq s\} \right\}$$

Recall: $f = Q(f,f,f)$ weakly, i.e,

$$\langle g, f\rangle = \langle g, Q(f,f,f)\rangle = \mathcal{Q}(g,f,f,f)$$

for all $g \in L^2(\mathbb{R})$.

Note

$$\alpha(s) = \sup\Big\{|\langle g, f\rangle| : \ \|g\| = 1, \text{supp}(g) \subset \{|x| \geq s\}\Big\}$$

It remains to estimate $\mathcal{Q}(g,f,f,f)$ uniformly in g.

Multi-linear estimates

Multi-linear estimates

The map $g, f \mapsto \mathcal{Q}(g, f, f, f)$ is non-linear (and non-local).

Multi-linear estimates

The map $g, f \mapsto \mathcal{Q}(g, f, f, f)$ is non-linear (and non-local).

The map $f_1, f_2, f_3, f_4 \mapsto \mathcal{Q}(f_1, f_2, f_3, f_4)$ is multi-linear.

Multi-linear estimates

The map $g, f \mapsto \mathcal{Q}(g, f, f, f)$ is non-linear (and non-local).

The map $f_1, f_2, f_3, f_4 \mapsto \mathcal{Q}(f_1, f_2, f_3, f_4)$ is multi-linear.

Fact 1: If g is supported in $\{|x| \geq s\}$ and f_1, f_2, f_3 are supported in $\{|x| \leq s/3\}$, then

$$\mathcal{Q}(g, f_1, f_2, f_3) = 0$$

Multi-linear estimates

The map $g, f \mapsto \mathcal{Q}(g, f, f, f)$ is non-linear (and non-local).

The map $f_1, f_2, f_3, f_4 \mapsto \mathcal{Q}(f_1, f_2, f_3, f_4)$ is multi-linear.

Fact 1: If g is supported in $\{|x| \geq s\}$ and f_1, f_2, f_3 are supported in $\{|x| \leq s/3\}$, then

$$\mathcal{Q}(g, f_1, f_2, f_3) = 0$$

So even though $\mathcal{Q}$ is non-local, it retains some locality (quasi-local).

First reduction

Let $g \in L^2$ with supp$(g) \subset \{|x| \geq s\}$.

First reduction

Let $g \in L^2$ with supp$(g) \subset \{|x| \geq s\}$.

Split $f = f_< + f_>$ with $f_< := \chi_{(-s/3,s/3)} f$ and $f_> = \chi_{\{|x|>s/3\}} f$.

First reduction

Let $g \in L^2$ with supp$(g) \subset \{|x| \geq s\}$.
Split $f = f_< + f_>$ with $f_< := \chi_{(-s/3,s/3)} f$ and $f_> = \chi_{\{|x|>s/3\}} f$.

$$\mathcal{Q}(g, f, f, f) = \mathcal{Q}(g, f_< + f_>, f_< + f_>, f_< + f_>)$$

First reduction

Let $g \in L^2$ with supp$(g) \subset \{|x| \geq s\}$.
Split $f = f_< + f_>$ with $f_< := \chi_{(-s/3,s/3)} f$ and $f_> = \chi_{\{|x|>s/3\}} f$.

$$\begin{aligned}\mathcal{Q}(g,f,f,f) &= \mathcal{Q}(g, f_< + f_>, f_< + f_>, f_< + f_>) \\ &= \mathcal{Q}(g, f_<, f_<, f_<) + \mathcal{Q}(g, f_>, f_>, f_>) \\ &\quad + \mathcal{Q}(g, f, f_<, f_>) + \mathcal{Q}(g, f_>, f, f_<) + \mathcal{Q}(g, f_<, f_>, f)\end{aligned}$$

First reduction

Let $g \in L^2$ with supp$(g) \subset \{|x| \geq s\}$.
Split $f = f_< + f_>$ with $f_< := \chi_{(-s/3,s/3)} f$ and $f_> = \chi_{\{|x|>s/3\}} f$.

$$\begin{aligned}
\mathcal{Q}(g,f,f,f) &= \mathcal{Q}(g, f_< + f_>, f_< + f_>, f_< + f_>) \\
&= \mathcal{Q}(g, f_<, f_<, f_<) + \mathcal{Q}(g, f_>, f_>, f_>) \\
&\quad + \mathcal{Q}(g, f, f_<, f_>) + \mathcal{Q}(g, f_>, f, f_<) + \mathcal{Q}(g, f_<, f_>, f) \\
&\overset{\text{Fact 1}}{=} \mathcal{Q}(g, f_>, f_>, f_>) \\
&\quad + \mathcal{Q}(g, f, f_<, f_>) + \mathcal{Q}(g, f_>, f, f_<) + \mathcal{Q}(g, f_<, f_>, f)
\end{aligned}$$

Multi-linear II

Multi-linear II

Fact 2: $|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim \|f_1\|\|f_2\|\|f_3\|\|f_4\|$

Multi-linear II

Fact 2: $|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim \|f_1\|\|f_2\|\|f_3\|\|f_4\|$

So

$$|\mathcal{Q}(g, f_>, f_>, f_>)| \lesssim \|g\|\|f_>\|^3 = \alpha(s/3)^3.$$

Multi-linear II

Fact 2: $|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim \|f_1\|\|f_2\|\|f_3\|\|f_4\|$

So

$$|\mathcal{Q}(g, f_>, f_>, f_>)| \lesssim \|g\|\|f_>\|^3 = \alpha(s/3)^3.$$

Remains to estimate terms of the form $\mathcal{Q}(g, f, f_<, f_>)$.

Multi-linear II

Fact 2: $|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim \|f_1\|\|f_2\|\|f_3\|\|f_4\|$

So

$$|\mathcal{Q}(g, f_>, f_>, f_>)| \lesssim \|g\|\|f_>\|^3 = \alpha(s/3)^3.$$

Remains to estimate terms of the form $\mathcal{Q}(g, f, f_<, f_>)$.

Note supp(g) and supp$(f_<)$ have distance $2s/3$.

Multi-linear II

Fact 2: $|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim \|f_1\| \|f_2\| \|f_3\| \|f_4\|$

So

$$|\mathcal{Q}(g, f_>, f_>, f_>)| \lesssim \|g\| \|f_>\|^3 = \alpha(s/3)^3.$$

Remains to estimate terms of the form $\mathcal{Q}(g, f, f_<, f_>)$.

Note supp(g) and supp$(f_<)$ have distance $2s/3$.

Fact 3:

$$|\mathcal{Q}(g, f, f_<, f_>)| \lesssim \frac{\|g\| \|f\| \|f_<\| \|f_>\|}{\sqrt{s}} \leq \frac{\alpha(0)^2 \alpha(s/3)}{\sqrt{s}}.$$

and similar for the other terms.

Multi-linear II

So

$$|\mathcal{Q}(g,f,f,f)| = |\mathcal{Q}(g,f_< + f_>,f_< + f_>,f_< + f_>)|$$

Multi-linear II

So

$$
\begin{aligned}
|\mathcal{Q}(g,f,f,f)| &= |\mathcal{Q}(g,f_< + f_>, f_< + f_>, f_< + f_>)| \\
&\leq |\mathcal{Q}(g,f_>,f_>,f_>)| \\
&\quad + |\mathcal{Q}(g,f,f_<,f_>)| + |\mathcal{Q}(g,f_>,f,f_<)| + |\mathcal{Q}(g,f_<,f_>,f)|
\end{aligned}
$$

Multi-linear II

So

$$
\begin{aligned}
|\mathcal{Q}(g,f,f,f)| &= |\mathcal{Q}(g, f_< + f_>, f_< + f_>, f_< + f_>)| \\
&\leq |\mathcal{Q}(g, f_>, f_>, f_>)| \\
&\quad + |\mathcal{Q}(g, f, f_<, f_>)| + |\mathcal{Q}(g, f_>, f, f_<)| + |\mathcal{Q}(g, f_<, f_>, f)| \\
&\lesssim \alpha(s/3)^3 + \frac{1}{\sqrt{s}}\alpha(0)^2\alpha(s/3).
\end{aligned}
$$

Multi-linear II

So

$$
\begin{aligned}
|\mathcal{Q}(g,f,f,f)| &= |\mathcal{Q}(g,f_< + f_>, f_< + f_>, f_< + f_>)| \\
&\leq |\mathcal{Q}(g,f_>,f_>,f_>)| \\
&\quad + |\mathcal{Q}(g,f,f_<,f_>)| + |\mathcal{Q}(g,f_>,f,f_<)| + |\mathcal{Q}(g,f_<,f_>,f)| \\
&\lesssim \alpha(s/3)^3 + \frac{1}{\sqrt{s}}\alpha(0)^2\alpha(s/3).
\end{aligned}
$$

Hence

$$\alpha(s) = \sup(|\langle g,f\rangle| : \ \|g\| = 1, \text{supp}(g) \subset \{|x| \geq s\} \,)$$

.

Multi-linear II

So

$$
\begin{aligned}
|\mathcal{Q}(g,f,f,f)| &= |\mathcal{Q}(g, f_< + f_>, f_< + f_>, f_< + f_>)| \\
&\leq |\mathcal{Q}(g, f_>, f_>, f_>)| \\
&\quad + |\mathcal{Q}(g, f, f_<, f_>)| + |\mathcal{Q}(g, f_>, f, f_<)| + |\mathcal{Q}(g, f_<, f_>, f)| \\
&\lesssim \alpha(s/3)^3 + \frac{1}{\sqrt{s}} \alpha(0)^2 \alpha(s/3).
\end{aligned}
$$

Hence

$$
\alpha(s) = \sup_g |\mathcal{Q}(g,f,f,f)|
$$

.

Multi-linear II

So

$$
\begin{aligned}
|\mathcal{Q}(g,f,f,f)| &= |\mathcal{Q}(g, f_< + f_>, f_< + f_>, f_< + f_>)| \\
&\leq |\mathcal{Q}(g, f_>, f_>, f_>)| \\
&\quad + |\mathcal{Q}(g, f, f_<, f_>)| + |\mathcal{Q}(g, f_>, f, f_<)| + |\mathcal{Q}(g, f_<, f_>, f)| \\
&\lesssim \alpha(s/3)^3 + \frac{1}{\sqrt{s}} \alpha(0)^2 \alpha(s/3).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\alpha(s) &= \sup_g |\mathcal{Q}(g,f,f,f)| \\
&\lesssim \alpha(s/3)^3 + \frac{1}{\sqrt{s}} \alpha(0)^2 \alpha(s/3).
\end{aligned}
$$

Why quasi-local?

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dx dt$$

Why quasi-local?

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dx dt$$

Now use

$$T_t f(x) = \frac{1}{\sqrt{4\pi i t}} \int_R e^{i(x-y)^2/(4t)} f(y)\, dy$$

multiply out everything like crazy and do the x-integration

Why quasi-local?

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dx dt$$

$$= \int_0^1 \delta(y_1 - y_2 + y_3 - y_4) e^{i(y_1^2 - y_2^2 + y_3^2 - y_4^2)/(4t)}$$

$$\overline{f_1(y_1)} f_2(y_2) \overline{f_3(y_3)} f_4(y_4)\, dy$$

Why quasi-local?

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dx dt$$

$$= \int_0^1 \delta(y_1 - y_2 + y_3 - y_4) e^{i(y_1^2 - y_2^2 + y_3^2 - y_4^2)/(4t)}$$

$$\overline{f_1(y_1)} f_2(y_2) \overline{f_3(y_3)} f_4(y_4)\, dy$$

$$= 0$$

Why quasi-local?

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t g(x)} T_t f_1(x) \overline{T_t f_2(x)} T_t f_3(x)\, dx dt$$

$$= \int_0^1 \delta(y_1 - y_2 + y_3 - y_4) e^{i(y_1^2 - y_2^2 + y_3^2 - y_4^2)/(4t)}$$

$$\overline{f_1(y_1)} f_2(y_2) \overline{f_3(y_3)} f_4(y_4)\, dy$$

$$= 0$$

if the supports of f_1, f_2, f_3 and f_4 are such that their product is zero whenever $y_1 - y_2 + y_3 - y_4 = 0$.

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| = |\int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dx dt|$$

$$
\begin{aligned}
|\mathcal{Q}(f_1, f_2, f_3, f_4)| &= |\int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dx dt| \\
&\leq \int_0^1 \int_{\mathbb{R}} \prod_{j=1}^{4} |T_t f_j(x)|\, dx dt
\end{aligned}
$$

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| = |\int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dxdt|$$

$$\leq \int_0^1 \int_{\mathbb{R}} \prod_{j=1}^{4} |T_t f_j(x)|\, dxdt$$

$$\leq \prod_{j=1}^{4} \Big(\int_0^1 \int_{\mathbb{R}} |T_t f_j(x)|^4\, dxdt \Big)^{1/4}$$

$$
\begin{aligned}
|\mathcal{Q}(f_1,f_2,f_3,f_4)| &= \left|\int_0^1\int_{\mathbb{R}} \overline{T_tf_1(x)}T_tf_2(x)\overline{T_tf_3(x)}T_tf_4(x)\,dxdt\right| \\
&\le \int_0^1\int_{\mathbb{R}} \prod_{j=1}^4 |T_tf_j(x)|\,dxdt \\
&\le \prod_{j=1}^4 \Big(\int_0^1\int_{\mathbb{R}} |T_tf_j(x)|^4\,dxdt\Big)^{1/4} \\
&= \prod_{j=1}^4 \big(\mathcal{Q}(f_j,f_j,f_j,f_j)\big)^{1/4}
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{Q}(f_1, f_2, f_3, f_4)| &= \Big| \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x)\, dxdt \Big| \\
&\leq \int_0^1 \int_{\mathbb{R}} \prod_{j=1}^{4} |T_t f_j(x)|\, dxdt \\
&\leq \prod_{j=1}^{4} \Big(\int_0^1 \int_{\mathbb{R}} |T_t f_j(x)|^4\, dxdt \Big)^{1/4} \\
&= \prod_{j=1}^{4} \big(\mathcal{Q}(f_j, f_j, f_j, f_j) \big)^{1/4} \leq S_1^3 \prod_{j=1}^{4} \|f_j\|
\end{aligned}
$$

Multi-linear enhancement

Why

$$|\mathcal{Q}(g, f, f_<, f_>)| \lesssim \frac{\|g\|\|f\|\|f_<\|\|f_>\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}), \text{supp}(\widehat{f_<}))}}$$

Multi-linear enhancement

Why

$$|\mathcal{Q}(g,f,f_<,f_>)| \lesssim \frac{\|g\|\|f\|\|f_<\|\|f_>\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}),\text{supp}(\widehat{f_<}))}}$$

By Cauchy-Schwarz

$$|\mathcal{Q}(g,f,f_<,f_>)| \leq \|T_t g T_t f_<\|_{L^2([0,1]\times\mathbb{R})}\|T_t f T_t f_>\|_{L^2([0,1]\times\mathbb{R})}$$

Multi-linear enhancement

Why

$$|\mathcal{Q}(g, f, f_<, f_>)| \lesssim \frac{\|g\| \|f\| \|f_<\| \|f_>\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}), \text{supp}(\widehat{f_<}))}}$$

By Cauchy-Schwarz

$$|\mathcal{Q}(g, f, f_<, f_>)| \leq \|T_t g T_t f_<\|_{L^2([0,1]\times\mathbb{R})} \|T_t f T_t f_>\|_{L^2([0,1]\times\mathbb{R})}$$

and

$$\|T_t f T_t f_>\|_{L^2([0,1]\times\mathbb{R})} \lesssim \|f\| \|f_>\|$$

Bi-linear Strichartz

Claim:

$$\|T_t g T_t f_<\|_{L^2([0,1]\times\mathbb{R})} \lesssim \frac{\|g\| \|f_<\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}), \text{supp}(\widehat{f_<}))}}$$

Bi-linear Strichartz

Claim:

$$\|T_t g T_t f_<\|_{L^2([0,1]\times\mathbb{R})} \lesssim \frac{\|g\|\,\|f_<\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}),\text{supp}(\widehat{f_<}))}}$$

Reason: The wave packets $T_t g$ and $T_t f_<$ have very different speed, hence their interaction should be small.

Bi-linear Strichartz

Claim:

$$\|T_t g T_t f_<\|_{L^2([0,1]\times\mathbb{R})} \lesssim \frac{\|g\|\|f_<\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}), \text{supp}(\widehat{f_<}))}}$$

Reason: The wave packets $T_t g$ and $T_t f_<$ have very different speed, hence their interaction should be small. This heuristic is made precise using arguments from harmonic analysis.

Bi-linear Strichartz

Claim:

$$\|T_t g T_t f_<\|_{L^2([0,1]\times\mathbb{R})} \lesssim \frac{\|g\|\,\|f_<\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}),\text{supp}(\widehat{f_<}))}}$$

Reason: The wave packets $T_t g$ and $T_t f_<$ have very different speed, hence their interaction should be small. This heuristic is made precise using arguments from harmonic analysis.

New insight: The same bound holds if the supports of g and $f_<$ are separated!

New Symmetry

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, t^{d-2}dtdx)} = 2^{1-d/2}\|T_\tau \check{f}_1 T_\tau \check{f}_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, d\tau dz)}$$

where $\check{f}_j$ is the inverse Fourier transform of f_j.

New Symmetry

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, t^{d-2}dtdx)} = 2^{1-d/2}\|T_\tau \check{f}_1 T_\tau \check{f}_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, d\tau dz)}$$

where $\check{f}_j$ is the inverse Fourier transform of f_j.

Consequence: All previously known bounds involving conditions on the Fourier transforms of f_1 and f_2 have direct x-space analogues.