

# On sharp Strichartz inequalities in low dimensions

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# Plan

- The Strichartz inequality
- Some History
- Proof of sharp Strichartz
  - The Representation Theorem
  - Classification of maximizers: how to use rotation invariance

# The free Schrödinger evolution

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$$i\partial_t u = -\Delta u \quad \text{on } L^2(\mathbb{R}^d)$$

With initial condition

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Thus

$$u \in L_t^\infty L_x^2 \quad \text{as a space-time function}$$

# Strichartz inequality

**Theorem** (Strichartz 1977). For  $p = p(d) = 2 + \frac{4}{d}$ ,

$$\|u\|_{L^p_{t,x}} = \left( \int_{\mathbb{R}} dt \int_{\mathbb{R}^d} dx |u(t, x)|^p \right)^{1/p} \leq S_d \|f\|_{L^2_x}$$

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the paraboloid  $\tau = k^2$  has positive Gaussian curvature in  $\mathbb{R}^{d+1}$ . Now apply the Stein-Tomas theorem.

# Some history

- Tomas 1975: Fourier restriction theorem for densities on the sphere
- Extended by Strichartz to some non-compact manifolds with non-vanishing Gaussian curvature.
- Much simplified proof by Ginibre and Velo 1985.
- Strichartz inequalities are at the heart of most studies of non-linear Schrödinger equations (from mid 1980 to now).

# Some questions

- What is

$$S_d = \sup_{f \neq 0} \frac{\|e^{it\Delta} f\|_{L^p_{t,x}}}{\|f\|_{L^2_x}} = ?$$

- Existence of maximizers: Are there  $f_* \in L^2(\mathbb{R}^d)$  with

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- What are all maximizers?
- all questions above turn out to be very hard to answer: The Strichartz inequality is **invariant** under all Galilei transformations (**translations** and **boosts**) and **scaling**, which is a *huge* non-compact group.

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- Milena Stanislavova 2004: for a related functional (the dispersion management functional) all maximizers are smooth.
- Damiano Foschi preprint 2004:  $S_1 = 12^{-1/12}$  and  $S_2 = 2^{-1/2}$  and Gaussians are (among the) maximizers.

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**Theorem** (100DM and Vadim Zharnitsky 2006). *Let  $f \in L^2(\mathbb{R}^d)$ .*

*a) If  $d = 1$  then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |e^{it\Delta} f(x)|^6 dx dt = \frac{1}{2\sqrt{3}} \langle f \otimes f \otimes f, P_1(f \otimes f \otimes f) \rangle_{L^2(\mathbb{R}^3)}$$

*b) If  $d = 2$  then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{it\Delta} f(x)|^4 dx dt = \frac{1}{4} \langle f \otimes f, P_2(f \otimes f) \rangle_{L^2(\mathbb{R}^4)}.$$

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$P_2$ : orthogonal projection onto functions in  $L^2(\mathbb{R}^4)$  invariant under rotations fixing the  $(1, 0, 1, 0)$  and  $(0, 1, 0, 1)$  directions.

# Consequences I

Note that  $\langle f \otimes f \otimes f, f \otimes f \otimes f \rangle_{L^2(\mathbb{R}^3)} = \|f\|_{L^2(\mathbb{R})}^6$ .

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i.e.,

$$S_1 \leq (2\sqrt{3})^{-1/6} = 12^{-1/12}.$$

and similarly,

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- If  $f(x) = e^{-ax^2}$ , then

$$f \otimes f \otimes f(\eta) = f(\eta_1)f(\eta_2)f(\eta_3) = e^{-a(\eta_1^2 + \eta_2^2 + \eta_3^2)} = e^{-a\eta^2}$$

is invariant under all rotations of  $\mathbb{R}^3$ .

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- In particular,  $S_1 = 12^{-1/12}$  and analogously  $S_2 = 2^{-1/2}$ .

# Classification of maximizers

**Theorem** (100DM and Vadim Zharnitsky). *a) If  $f \in L^2(\mathbb{R})$  is not identically zero and  $f \otimes f \otimes f$  is invariant under rotations of  $\mathbb{R}^3$  which keep the  $(1, 1, 1)$  direction fixed, then there exist  $A \in \mathbb{C}$ ,  $\mu \in \mathbb{R}$ ,  $\lambda > 0$ , and  $b \in \mathbb{C}$  such that*

$$f(x) = Ae^{(-\lambda+i\mu)x^2+bx}.$$

*b) If  $f \in L^2(\mathbb{R}^2)$  and  $f \otimes f$  is invariant under rotations of  $\mathbb{R}^4$  fixing  $(1, 0, 1, 0)$  and  $(0, 1, 0, 1)$  directions, then*

$$f(x) = Ae^{(-\lambda+i\mu)|x|^2+b \cdot x}$$

*for some  $A \in \mathbb{C}$ ,  $\mu \in \mathbb{R}$ ,  $\lambda > 0$ , and  $b \in \mathbb{C}^2$*

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then

$$F(v) = e^{-av^2}.$$

That is,  $F$  is a centered Gaussian.

# Proof of the Representation Formula $d = 1$

Have

$$u(t, x) = T_t f(x) = \frac{1}{\sqrt{4\pi it}} \int_{\mathbb{R}} e^{i(x-y)^2/(4t)} f(y) dy$$

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$$u(t, x)^3 = \frac{1}{\sqrt{4\pi it}^3} \left( \int_{\mathbb{R}} e^{i(x-y)^2/(4t)} f(y) dy \right)^3$$

and

$$|u(t, x)|^6 = \frac{1}{(4\pi |t|)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-ix(\eta_1 + \eta_2 + \eta_3 - \zeta_1 - \zeta_2 - \zeta_3)/(2t)} e^{-i(|\eta|^2 - |\zeta|^2)/(4t)} \overline{f(\eta_1) f(\eta_2) f(\eta_3)} f(\zeta_1) f(\zeta_2) f(\zeta_3) d\eta d\zeta$$

Now do the  $x$ -integration with the change of variables

$x = 2tz$ , so  $dx = 2|t|dz$ , using  $\delta(\beta) = \frac{1}{2\pi} \int e^{ix\beta} dx$ ,

$$\int_{\mathbb{R}} |u(t, x)|^6 dx =$$
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Then do the  $t$ -integration with  $t = 1/(4\tau)$ , so  $dt = 4t^2 d\tau$ ,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u(t, x)|^6 dx dt = \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta((1, 1, 1)(\eta - \zeta)) \delta(|\eta|^2 - |\zeta|^2) \overline{f \otimes f \otimes f(\eta) f \otimes f \otimes f(\zeta)} d\eta d\zeta$$

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$$q(F, G) := \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta((1, 1, 1)(\eta - \zeta)) \delta(|\eta|^2 - |\zeta|^2) \overline{F(\eta)} G(\zeta) d\eta d\zeta$$

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Thus any bound on  $q$  yields a bound on  $S_1^6$ .

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**Claim:**

$$A = \frac{1}{2\sqrt{3}} P_1$$

where  $P_1 : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^2)$  is the orthogonal projection operator mapping into functions  $F$  which are invariant under rotations of  $\mathbb{R}^3$  around the  $(1, 1, 1)$  direction.

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is invariant under rotations of  $\mathbb{R}^3$  which keep  $(1, 1, 1)$  fixed.

**Fact 1:**

$$A : L^2(\mathbb{R}^3) \rightarrow \text{Range}(P_1)$$

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**Fact 2:**

$$2\sqrt{3}A \text{ is the identity on } \text{Range}(P_1)$$

Indeed, put  $\tau = (1, 1, 1)\eta$  and  $\xi = |\eta|^2$  and apply  $2\sqrt{3}A$  to  $\tilde{G}((1, 1, 1)\zeta, |\zeta|^2)$ .

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 &= \int_0^\infty \tilde{G}(\tau, r^2 + \tau^2/3) \delta(\xi - \tau^2/3 - r^2) d(r^2) \\
 &= \tilde{G}(\tau, \xi) = \tilde{G}((1, 1, 1)\eta, |\eta|^2)
 \end{aligned}$$

# Gaussians and rotations

# Gaussians and rotations

Toy problem: Assume that for  $f \in L^2(\mathbb{R})$  the function

$$h = f \otimes f : \mathbb{R}^2 \rightarrow \mathbb{C}$$

$$(x, y) \mapsto h(x, y) := f(x)f(y)$$

is invariant under rotations of  $\mathbb{R}^2$ . Then  $f$  is a centered Gaussian.

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$$0 = Lh = xf(x)f'(y) - yf'(x)f(y)$$

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Thus  $(\log(f))' = cx$  and  $f$  is a centered Gauss.

# Step 2

Let  $P_t(z) = \frac{1}{2\pi t} e^{-(z)^2/(2t)}$  for  $z \in \mathbb{R}^2$  and  $Q_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$  for  $x \in \mathbb{R}$ .

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$$P_t * h = Q_t * f \otimes Q_t * f$$

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If  $f \in L^2(\mathbb{R})$  is not indentially zero and  $h = f \otimes f$  is invariant under rotations of  $\mathbb{R}^2$ , then  $Q_t * f$  never vanishes.

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In particular,  $Q_t * f$  never vanishes!