RESEARCH STATEMENT

DIMITRIOS KOUKOULOPOULOS

I am interested in analytic and probabilistic number theory. In my thesis I studied primarily questions on the distribution of divisors of integers. I am also interested in the theory of L-functions and its applications on prime number theory as well as sieve methods.

1. Summary of results

The main motivating problem for my thesis is the *Erdős multiplication table problem* posed in 1955 (see [4], [5]): Given a large number N, how many distinct products of the form n_1n_2 with $n_1 \le N$ and $n_2 \le N$ are there? Call $A(N)$ the quantity in question. Despite the elementary nature of this question, progress on bounding $A(N)$ has been very slow. The main reason is that $A(N)$ is intimately connected with deep questions about the distribution of divisors of integers. More precisely, if we let $H(x,y,z)$ denote the counting function of integers with a divisor in $(y,z]$, then in order to bound $A(N)$ it suffices to obtain estimates on $H(x,y,z)$ when $z=2y$. In [7] Ford, improving upon work of Tenenbaum [19], established the order of $H(x,y,z)$ for all x,y,z and, as a consequence, the order of $A(N)$.

In my thesis I worked towards extending Ford's results in two different directions. The first one is the k-dimensional multiplication table problem: we seek bounds on $A_k(N)$, the number of integers n that can be written as product $n = n_1 \cdots n_k$ with $n_i \le N$ for all $i \in \{1,\dots,k\}$. In an analogous way to the case $k=2$, the number $A_k(N)$ is connected with the counting function of localized factorizations of integers: Given $x \ge 1$, $\boldsymbol{y} = (y_1,\dots,y_{k-1}) \in [0,+\infty)^{k-1}$ and $\boldsymbol{z} = (z_1,\dots,z_{k-1}) \in [0,+\infty)^{k-1}$ define

$$H^{(k)}(x,\boldsymbol{y},\boldsymbol{z}) = |\{n \le x : \exists\, d_1 \cdots d_{k-1} | n \text{ with } y_i < d_i \le z_i \ (1 \le i \le k-1)\}|.$$

In [16] I established the order of magnitude of $H^{(k)}(x,\boldsymbol{y},2\boldsymbol{y})$ for all $k \ge 3$ when the numbers $\log y_1,\dots,\log y_{k-1}$ are of the same order. As a consequence, I determined the order of $A_k(N)$ for $k \ge 3$. Combining the results of [7] for $k=2$ and of [16] for $k \ge 3$ we have the following estimate.

Theorem 1 (Ford [7], K. [16]). *Let $k \geq 2$. For every $N \geq 3$ we have that*

$$A_k(N) \asymp_k \frac{N^k}{(\log N)^{\delta_k}(\log\log N)^{3/2}},$$

where

$$\delta_k = \int_1^{\frac{k-1}{\log k}} \log t \, dt.$$

(Here $f \asymp_a g$ means that $M^{-1}f \leq g \leq Mf$ for some constant M which depends on a only.) The main difficulties in proving Theorem 1 for $k \geq 3$ lie in the lower bound argument because of the presence of certain complicated combinatorics.

In [14], Haynes and Homma used the methods developed by Ford to study a related problem. Given $R \geq 1$ denote with $\mathcal{F}_R$ the set of Farey fractions with denominator at most R, that is

$$\mathcal{F}_R = \left\{ \frac{a}{b} : 1 \leq a \leq b \leq R, \ (a,b) = 1 \right\}.$$

Then a natural question is to estimate the cardinality of the set

$$\mathcal{F}_R(k) = \underbrace{\mathcal{F}_R + \cdots + \mathcal{F}_R}_{k \text{ times}} \pmod 1.$$

In their paper [14], they established the order of $|\mathcal{F}_R(2)|$. Using bounds on $H^{(k)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ I extended their result to $|\mathcal{F}_R(k)|$ for $k \geq 3$. Combining the results of [14] and [16] we have the following theorem.

Theorem 2 (Haynes and Homma [14], K. [16]). *Let $k \geq 2$. For every $R \geq 3$ we have that*

$$|\mathcal{F}_R(k)| \asymp \frac{R^{2k}}{(\log R)^{\delta_k}(\log\log R)^{3/2}}.$$

In [15] I took a different path and studied a special case of the *restricted multiplication table problem*: Given $\mathscr{B} \subset \mathbb{N}$ estimate

$$A(N; \mathscr{B}) = |\{n_1 n_2 \in \mathscr{B} : n_i \leq N \ (1 \leq i \leq 2)\}|.$$

Again, in order to solve this problem it is sufficient to bound

$$H(x, y, z; \mathscr{B}) = |\{n \in [1, x] \cap \mathscr{B} : \exists\, d | n \text{ with } y < d \leq z\}|,$$

when $z = 2y$. If $\mathscr{B}$ is reasonably well-distributed in arithmetic progressions, then we expect that

$$H(x, y, z; \mathscr{B}) \approx \frac{|\mathscr{B} \cap [1, x]|}{x} H(x, y, z).$$

In [7] Ford focused on the special and important case when $\mathscr{B} = P_s = \{p+s : p \text{ prime}\}$ for some fixed $s \neq 0$. He obtained upper bounds of the expected order on $H(x,y,z;P_s)$ for all x,y,z as well as lower bounds when y and z are fixed powers of x. In [15] I worked towards supplying the lower bounds in the broadest possible range of the parameters y and z. One of the results in this paper establishes the desired lower bound when $3 \le y+1 \le z \le x$, provided that $z \ge y + y(\log y)^{-A}$ for some fixed $A \ge 0$. As a consequence we have the following theorem.

Theorem 3 (Ford [7], K. [15]). *Fix* $s \in \mathbb{Z} \setminus \{0\}$. *For* $N \ge 3$ *we have that*

$$A(N;P_s) \asymp_s \frac{A(N)}{\log N}.$$

In order to prove lower bounds on $A(N;P_s)$ we need fairly precise estimates of primes in arithmetic progressions, a problem for which our knowledge is limited. Specifically, we have to understand $H(x,y,2y;P_s)$ when y is very close to $\sqrt{x}$, which in turn depends heavily on estimates *on average* for primes in arithmetic progressions of modulus $d \le \sqrt{x}$ (actually, for technical reasons, we need to go up to $x^{1/2+\epsilon}$ for an arbitrarily small, but nevertheless fixed, positive ϵ). Therefore the Bombieri-Vinogradov theorem cannot handle this problem. To prove the lower bound implicit in Theorem 3, I used a result by Bombieri, Friedlander and Iwaniec [1, Theorem 9] along with an application of the fundamental lemma of sieve methods [9, Lemma 5] in order to smoothen certain sums and make [1, Theorem 9] applicable.

2. Future projects

After proving Theorem 1, I turned to the *generalized multiplication table problem*: Given $N_1,\dots,N_k$ estimate $A_k(N_1,\dots,N_k)$, the number of integers n that may be written as a product $n = n_1 \cdots n_k$ with $n_i \le N_i$ for all $i \in \{1,\dots,k\}$. The size of $A_k(N_1,\dots,N_k)$ is extremely sensitive to the relative size of

$$\ell_i = \log \frac{\log 3N_i}{\log N_{i-1}} \quad (1 \le i \le k-1),$$

where for simplicity we assume that $1 = N_0 \le N_1 \le \cdots \le N_k$. I have obtained some partial results on this more general problem. In particular, I determined the order of $A_3(N_1,N_2,N_3)$ in all cases.

Theorem 4 (K. [17]). *Let* $3 \le N_1 \le N_2 \le N_3$. *Then*

$$\frac{A_3(N_1,N_2,N_3)}{N_1N_2N_3} \asymp \frac{\ell_1\ell_2}{(\ell_1+\ell_2)^{5/2}} \exp\{-Q(3^\alpha)\ell_1 - Q(2^\alpha)\ell_2\},$$

where $Q(u)=\int_1^u \log t\,dt$ *and* α *is defined implicitly via the equation*

$$3^\alpha \log 3\ell_1 + 2^\alpha \log 2\ell_2 = 2\ell_1 + \ell_2.$$

This is still work in progress and my ultimate goal is to extend the above theorem to $k \geq 4$ in the broadest possible range of the parameters $N_1, \ldots, N_k$.

Furthermore, I am particularly interested in the circle of problems that are related to Siegel zeros and exceptional characters. Specifically, I am studying a series of papers by Iwaniec and Friedlander where, under the assumption that there is an exceptional character, they obtain asymptotics for the number of prime numbers in very short intervals [10] and in arithmetic progressions to large moduli [11]. The remarkable feature of their work is that such asymptotics are beyond the reach of the Generalized Riemann Hypothesis. In a different direction, Conrey and Iwaniec [2] showed how the existence of an exceptional character could distort the conjectural distribution of zeros of the Riemann ζ function on Montgomery's Pair Correlation Conjecture [18]. An interesting question would be whether one can use similar ideas to deduce more precise estimates on the pair correlation of zeros of the Riemann ζ function under the assumption of the Riemann hypothesis and the existence of an exceptional character.

Another area of number theory that is particularly appealing to me is sieve methods. I recently became aware of a new proof of Linnik's theorem by Iwaniec and Friedlander [8, Chapter 3] in which zero density estimates for Dirichlet L-functions are substituted by a parity-sensitive version of the sieve. I plan to study their method and try to apply it to other related problems.

References

1. E. Bombieri, J. B. Friedlander and H. Iwaniec, *Primes in arithmetic progressions to large moduli,* Acta Math. **156** (1986), no. 3-4, 203-251.
2. B. Conrey and H. Iwaniec, *Spacing of zeros of Hecke L-functions and the class number problem,* Acta Arith. **103** (2002), no. 3, 259-312.
3. H. Davenport, *Multiplicative Number Theory*, third. ed., Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 2000, Revised and with a preface by Hugh L. Montgomery.
4. P. Erdős, *Some remarks on number theory*, Riveon Lematematika **9** (1955), 45-48, (Hebrew. English summary).
5. P. Erdős, *An asymptotic inequality in the theory of numbers*, Vestnik Leningrad Univ. **15** (1960), no. 13, 41-49, (Russian).

6. K. Ford, *Integers with a divisor in* $(y, 2y]$, Anatomy of integers (Jean-Marie Koninck, Andrew Granville, and Florian Luca, eds.) CRM Proc. and Lect. Notes 46, Amer. Math. Soc., Providence, RI, 2008, 65-81.
7. K. Ford, *The distribution of integers with a divisor in a given interval,* Annals of Math. (2) **168** (2008), 367-433.
8. J. Friedlander, D. R. Heath-Brown, H. Iwaniec and J. Kaczorowski, *Analytic number theory. Lectures from the C.I.M.E. Summer School held in Cetraro, July 11–18, 2002,* Edited by A. Perelli and C. Viola. Lecture Notes in Mathematics, 1891. Springer-Verlag, Berlin; Fondazione C.I.M.E., Florence, 2006.
9. J. Friedlander and H. Iwaniec, *On Bombieri's asymptotic sieve,* Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) **5** (1978), 719-756.
10. J. Friedlander and H. Iwaniec, *Exceptional characters and prime numbers in short intervals,* Selecta Math. (N.S.) **10** (2004), no. 1, 61-69.
11. J. Friedlander and H. Iwaniec, *Exceptional characters and prime numbers in arithmetic progressions,* Int. Math. Res. Not. 2003, no. 37, 2033-2050.
12. H. Halberstam and H.-E. Richert, *Sieve methods,* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London-New York, 1974, London Mathematical Society Monographs, No. 4.
13. R. R. Hall and G. Tenenbaum, *Divisors,* Cambridge Tracts in Mathematics, vol. 90, Cambridge University Press, Cambridge, 1988.
14. A. K. Haynes and K. Homma, *The group ring* $\mathbb{Q}/\mathbb{Z}$ *and an application of a divisor problem,* Proc. Amer. Math. Soc. **137** (2009), 1285-1293.
15. D. Koukoulopoulos, *Divisors of shifted primes,* to appear in Int. Math. Res. Not. (2009).
16. D. Koukoulopoulos, *Localized factorizations of integers,* to appear in Proc. London Math. Soc. (2009).
17. D. Koukoulopoulos, *On the number of integers in a generalized multiplication table,* in preparation.
18. H. L. Montgomery, *The pair correlation of zeros of the zeta function,* Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 181-193. Amer. Math. Soc., Providence, R.I., 1973.
19. G. Tenenbaum, *Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné,* Compositio Math. 51 (1984), 243-263.