Comments on [5], [14], [22], [30], and [31]

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During the two decades after the publication of the pioneering papers *Proof of the ergodic theorem* and *Proof of the quasi-ergodic hypothesis* by Birkhoff [Bi] and von Neumann [vN], but before the publication of *A general ergodic theorem* by Calderón in 1953, there were new proofs and generalizations of the results of Birkhoff and von Neumann given by many mathematicians. These include, among others, Khintchine, Hopf, Kolmogorov, F. Riesz, Yoshida, Kakutani, Wiener, Dunford, Pitt, Doob, and Zygmund. Much of this substantial body of work is referenced in [Ho2], [Ka], and [Kr]. Explorations in many new directions have continued since then and have greatly expanded the connections and applications of ergodic theory to other parts of mathematics and beyond. Calderón's 1953 paper [5] played an important role in this development.

In the paper, Calderón studies the behavior of averages of the form

$$\frac{1}{|N_t|}\int_{N_t} F(gx)\,dg. \tag{1}$$

Here $(N_t)_{t>0}$ is a family of compact symmetric neighborhoods of the identity of a locally compact group G of one-to-one measure-preserving transformations g on a measure space E of finite measure, $|N_t|$ is the left invariant Haar measure of N_t with dg denoting an element of this measure, and F is an integrable function on E such that $(g, x) \mapsto F(gx)$ has appropriate measurability. It is not assumed that G is abelian but it is assumed that G is ergodic in the sense that there exists a family $(N_t)_{t>0}$ as above satisfying $N_t N_s \subset N_{t+s}$ and $|N_{2t}| < \alpha |N_t|$ with the choice of the constant α not depending on t. Calderón proves a pointwise ergodic theorem for the averages (1) after first proving some of the properties of ergodic groups, a mean ergodic theorem, a maximal ergodic theorem, and a dominated ergodic theorem. He then shows how these theorems carry over to the case in which the ergodic group G is replaced by the product of several ergodic groups.

In this 1953 paper, Calderón significantly extended the earlier work of Wiener [Wi], who considered the cases $G = \mathbf{Z}^n$ and $G = \mathbf{R}^n$, and Dunford [Du] and Zygmund [Zy], who considered the products of finitely many noncommutative transformations. In turn, the 1953 paper of Calderón led to interesting new developments, one being Tempelman's ergodic theorem for amenable groups. See [Te1] and, and for more, his book [Te2] and the references given there.

One of the first major papers on ergodic theory to appear after Calderón's 1953 paper was Cotlar's paper [Co] giving *A unified theory of Hilbert transforms and ergodic theory.* It had been felt for a number of years that there were too many analogies between the properties of the Hilbert transform and the properties of ergodic averages, for example their similar L^p behavior, to be a mere coincidence. But it was realized that the connection would not be entirely simple since some of the behavior was not similar, for example ergodic averages take positive functions to positive functions but the Hilbert transform does not. Using some of his earlier work, also published in Volume 1 of Revista, Cotlar develops a framework that allows him to prove many of the standard properties of both the Hilbert transform and ergodic averages. The Calderón 1968 paper [22] entitled *Ergodic theory and translation-invariant operators* casts new light on this connection. Calderón shows, among other things, that the maximal ergodic theorem is "an immediate consequence of the maximal theorem of Hardy and Littlewood," and that the results of Cotlar for the *ergodic Hilbert transform* follow immediately from the corresponding results for the classical Hilbert transform.

In this 1968 paper, Calderón formulates a transference principle, nowadays often referred to as the Calderón Transference Principle, that has been an important influence on later work in ergodic theory and harmonic analysis. Coifman and Weiss [CW] discuss some of the newer applications of transference and Bellow [Be] does the same for some earlier as well as later work. She illustrates the Calderón Transference Principle with the following special case (also see, for example, [Ak], [J1], and [J2]).

For each nonnegative integer n, let $\mu_n\varphi$ denote the convolution of $\varphi \in \ell^1(\mathbf{Z})$ and a probabilty measure μ_n on the σ-algebra of all subsets of $\mathbf{Z}$. Let $(X, \mathcal{A}, m)$ be a probability space and $\tau : X \to X$ an invertible measure-preserving transformation. Define the function $\mu_n f$ on X for each $f \in L^1(X)$ by

$$(\mu_n f)(x) = \sum_{j \in \mathbf{Z}} \mu_n(j) f(\tau^j x).$$

(i) Is there a number $C_1 \in (0, \infty)$ such that, for all $\varphi \in \ell^1_+(\mathbf{Z})$ and $\lambda > 0$,

$$\# \left\{ j \in \mathbf{Z} : \sup_{n \geq 0} (\mu_n \varphi)(j) > \lambda \right\} \leq \frac{C_1}{\lambda} \|\varphi\|_{\ell^1(\mathbf{Z})}?$$

(ii) Is there a number $C_2 \in (0, \infty)$ such that, for all $(X, \mathcal{A}, m)$ and τ as above and all $f \in L^1_+(X)$ and $\lambda > 0$,

$$m \left\{ x \in X : \sup_{n \geq 0} (\mu_n f)(x) > \lambda \right\} \leq \frac{C_2}{\lambda} \|f\|_{L^1(X)}?$$

The answer for each question is of course "It depends on the sequence (μ_n)." Consider the following condition on (μ_n): there are numbers $\alpha \in (0, 1]$ and $C \in (0, \infty)$ such that, for all $n \geq 0$ and all x and y in $\mathbf{Z}$ satisfying $0 < 2|y| \leq |x|$,

$$|\mu_n(x + y) - \mu_n(x)| \leq C \frac{|y|^\alpha}{|x|^{1+\alpha}}. \tag{2}$$

Under this condition, Zo [Zo] and, independently, Bellow and Calderón [31] prove that C_1 exists. Bellow and Calderón also observe in this 1999 paper that if C_1 exists, then so does C_2, a simple application of the Calderón Transference Principle. One of the main results of their paper is that if μ is a strictly aperiodic probability measure on $\mathbf{Z}$ such that $\sum_k k^2 \mu(k) < \infty$ and $\sum_k k\mu(k) = 0$, then the sequence (μ^n) of nth convolutions of μ with itself satisfies (2).

In their 1962 paper [14] entitled *Convolution operators on Banach space valued functions*, Benedek, Calderón, and Panzone prove some inequalities for convolution operators that, among other applications, generalize to several variables some of the classical inequalities for the Hilbert transform and the function g of Littlewood and Paley. Reference is made to the earlier work of Cotlar [Co] and Hörmander [Hö], who give further references. There is also the contemporaneous work of Schwartz [Sc].

The interpolation results of the Benedek, Calderón, Panzone paper hold for all Banach spaces but, later in the paper, they focus on convolution operators for Hilbert space valued functions because some of the later theorems do not hold for all Banach spaces. In [Bo], Bourgain uses a different approach and obtains a result that holds for all UMD spaces.

A Banach space B is a UMD space if for some p in the interval $(1, \infty)$ there is a positive real number β_p such that

$$\left\| \sum_{k=0}^{n} \varepsilon_k d_k \right\|_{L^p([0,1],B)} \leq \beta_p \left\| \sum_{k=0}^{n} d_k \right\|_{L^p([0,1],B)}$$

for all B-valued martingale difference sequences $(d_k)_{k \geq 0}$, all sequences $(\varepsilon_k)_{k \geq 0}$ of numbers in $\{1, -1\}$, and all positive integers n; equivalently, B is UMD if all martingale difference sequences in $L^p([0,1], B)$ are unconditional. The Lebesgue probability space on the unit interval can be replaced by any other nonatomic probability space. This definition does not depend on the choice of p in

$(1, \infty)$ since B is a UMD space for a particular p if and only if there is a real-valued biconvex function ζ on $B \times B$ such that $\zeta(0,0) > 0$ and

$$\zeta(x,y) \leq \|x+y\| \text{ if } \|\mathrm{x}\| = \|\mathrm{y}\| = 1,$$

a characterization of UMD spaces that is independent of p. Among other characterizations of UMD spaces, the following is perhaps the most important: B is UMD if and only if for some $p \in (1, \infty)$, and hence for all such p, the Hilbert transform is a bounded operator on $L^p(\mathbf{R}, B)$. Among the UMD Banach spaces are the classical reflexive Lebesgue spaces, the reflexive Orlicz spaces, the reflexive trace-class spaces, and the reflexive non-commutative $L^p(M, \tau)$-spaces where M is a von Neumann algebra with a faithful, normal, semifinite trace τ. For some of the history of UMD spaces and more on their properties, see [Bu] and the references cited there.

Bourgain's theorem states that if $1 < p < \infty$ and B is a UMD space with a normalized unconditional basis $(e_j)_{j \geq 0}$, then any operator $T : L^p(\mathbf{R}^n, B) \to L^p(\mathbf{R}^n, B)$ defined by

$$T(\sum_{k=0}^{\infty} f_j e_j) = \sum_{k=0}^{\infty} T_j(f_j) e_j \tag{3}$$

is bounded. Here $f_j \in L^p(\mathbf{R}^n)$ and T_j is a Calderón-Zygmund singular integral operator with a convolution kernel. The operator T is first defined on appropriate subspaces of $L^p(\mathbf{R}^n, B)$ and the family of norms of T on these subspaces is shown to be bounded, after which the extension to the whole space is straightforward, proving (3).

In their 1995 paper [30], Bellow, Calderón, and Krengel consider a family of operators T_t studied by Eberhard Hopf [Ho1], who was interested in the limiting distribution as $t \uparrow \infty$ of a cloud of particles in a box. Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mu})$ be the product of the probability space $(\Omega, \mathcal{F}, \mu)$ and the Lebesgue measure space $([0, \infty), \mathcal{L}, \lambda)$. Let $(\tau_t)_{t \in \mathbf{R}}$ be a measurable measure-preserving flow on $(\Omega, \mathcal{F}, \mu)$. Fix $h \in \text{Ł}^{\infty}(\widetilde{\mu})$. For each $t \in [0, \infty)$, let $T_t : L^1(\widetilde{\mu}) \to L^1(\mu)$ be defined by

$$(T_t f)(\omega) = \int_0^{\infty} f(\tau_{ts}\omega, s) h(\omega, s)\, d\lambda(s)$$

Hopf proved that $T_t f$ converges in $L^1(\mu)$-norm as $t \uparrow \infty$. He also conjectured that $T_t f$ converges almost everywhere. Bellow and Krengel [BK] give a counterexample to this conjecture and, in addition, prove that $T_t f$ does not always converge almost everywhere as $t \downarrow 0$. In their 1995 paper, Bellow, Calderón, and Krengel strengthen this result by proving that almost-everywhere convergence can fail to hold in a dramatic way for $T_t f$ as $t \uparrow \infty$, and also as $t \downarrow 0$: using the Perron tree construction, they give an example for which $(T_t)_{t \geq 0}$ satisfies the "strong sweeping out property." In the example, $(\Omega, \mathcal{F}, \mu)$ is the probabilty space with $\Omega = [0, 1)$ and μ Lebesgue measure. The flow is given by $\tau_t(\omega) = \omega + t \pmod 1$, and h is given by $h(\omega, s) = 1$ for $s \in [0, 1)$ and $h(\omega, s) = 0$ for $s \in [1, \infty)$. They prove that for each $\epsilon > 0$ there is a function $f \in L^1(\widetilde{\mu})$, the characteristic function of a set E that satisfies $\widetilde{\mu}(E) \leq \epsilon$, such that almost everywhere $\limsup_{t \uparrow \infty} T_t f = 1$ and $\liminf_{t \uparrow \infty} T_t f = 0$. They also prove the parallel result with $t \downarrow 0$.

Both [30] and [31] indicate that Calderón was interested in ergodic theory to the end of his career. His contributions to ergodic theory during the last five decades have had a major and continuing influence on the work of many mathematicians. Here are just a few more examples. In [NS], Nevo and Stein study the ergodic theory of free groups and prove a far-reaching generalization of Birkhoff's pointwise ergodic theorem. Also, see [Ne] and the references given there. In [Li], Elon Lindenstrauss is able to penetrate deeply into the mysteries of Fϕlner sequences making it possible for him to prove some striking results on the ergodic theory of amenable group actions.

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