

INTERSECTION MULTIPLICITIES, CANONICAL ELEMENT CONJECTURE AND THE SYZYGY PROBLEM

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In honor of Melvin Hochster on the occasion of his 65th birthday.

Introduction.

In this article we are going to concentrate on the canonical element conjecture due to M. Hochster as well as several of its ramifications. In [17] Hochster introduced a number of equivalent forms of this conjecture and proved it in the equicharacteristic case. One of the earliest forms, the direct summand conjecture, was proved by Hochster [15] a decade earlier under the same hypothesis (see also [16]). In 1980 Evans and Griffith [10] gave an affirmative answer to the syzygy problem for equicharacteristic local rings. In the course of their proof, they implicitly established a result for finite free complexes [10] that Hochster explicitly isolated in his article [17]. He referred to the new result as the “improved new intersection theorem” (henceforth INIT) since it is clear that INIT implies the new intersection theorem ([17]). Of course INIT remains a “conjecture” in the case of mixed characteristic. In the same article [17] Hochster pointed out that INIT was a consequence of the canonical element conjecture and later the first author [3] showed that the two conjectures are equivalent. Over the years several special cases of the canonical element conjecture have been proved and new equivalent forms have been introduced ([2], [3], [5], [6], [7], [8], [14]). The four main equivalent versions of this conjecture, i.e., the direct summand conjecture, the monomial conjecture, the canonical element conjecture and the

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improved new intersection conjecture along with the statement of the syzygy problem are given at the end of our introduction.

In section 1 our main focus is on a formulation of the monomial conjecture in terms of comparison of lengths “ Tor_0 ” and “ Tor_1 ” of a pair of finitely generated modules over a regular local ring. Given a local ring A and a pair of finitely generated modules M and N such that $\text{length}(M \otimes_A N) < \infty$ (henceforth “ ℓ ” will denote length), we raise the following question (Q): Is $\ell(M \otimes_A N) > \ell(\text{Tor}_1^A(M, N))$? It is clear that (Q) has obvious negative answers, e.g., when $M = N = K$, the residue field of A , or when $M = K$, $N = \mathfrak{m}$, the maximal ideal of A . To get to the heart of the matter for A , a regular local ring, we first observe that (Q) boils down to the following: Is $\chi^A(M, N) > \chi_2^A(M, N)$? See section 1 for definitions of χ and χ_i . In Proposition 1.1(b) we argue, if A is an equicharacteristic or unramified regular local ring, then (Q) has a negative answer should $\dim M + \dim N < \dim A$ (i.e., when vanishing holds) and $\text{depth } M + \text{depth } N < \dim A - 1$. If $\dim M + \dim N = \dim A$, then Proposition 1.1(c) gives a positive answer should M or N be a perfect module and the other module satisfies “ $\dim - \text{depth} = 1$.” Even when $N = A/I$ and I is generated by an A -sequence (the best scenario for both χ and χ_i) we are unable to get a definite answer for (Q) when $\dim A/I + \dim M = \dim A$. The best we can prove in this situation is the following:

Theorem (1.4). *Let (A, \mathfrak{m}) be a local ring, I an ideal of A generated by an A -sequence and let M be a finitely generated A -module such that $\ell(M/IM) < \infty$, and $\dim A/I + \dim M = \dim A$. Then there exists an integer $t > 0$ and a minimal set of generators x_1, \dots, x_n of I such that (Q) has a positive answer for the pair $(A/I_{n,s}, M)$, where $I_{n,s} = (x_1, \dots, x_{n-1}, x_n^s)$, for $s \geq t$.*

Our proof of this theorem requires a) an extension of definition and properties of superficial elements, introduced by Samuel [25] from local rings to finitely generated modules over local rings and b) several properties of Hilbert-Samuel multiplicity (see Lemma (1.2) and Proposition (1.3)).

Our next theorem connects a positive answer for (Q) in a special case with an affirmative answer for the monomial conjecture.

Theorem (1.5). *The monomial conjecture is valid for all local rings if and only if, for every unramified and equicharacteristic regular local ring A and for every pair of ideals I, J of A such that i) I is a complete intersection ideal, ii) J is an almost complete intersection ideal, iii) $ht I + ht J = \dim A$ and iv) $I + J$ is m -primary, (Q) has a positive answer for the pair $(A/I, A/J)$.*

The above theorem demonstrates in a direct way why a definitive answer to (Q) , even in the best—but non-obvious case, is difficult to come by.

As a follow-up of this theorem, we prove in our next theorem (Theorem (1.6)) that in order to prove the monomial conjecture it is enough to have a positive answer for (Q) when $(I + J)$ is not a complete intersection ideal in A .

In our final result of this section we describe the cases for which we can assert a positive answer for (Q) .

Theorem (1.7). *Let A, I, J be as in Theorem (1.5) satisfying conditions i), ii), iii) and iv). Then (Q) has a positive answer for the pair $(A/I, A/J)$ in the following cases:*

- a) $\text{Tor}_1^A(A/I, A/J)$ is cyclic,
- b) $\text{Tor}_1^A(A/I, A/J)$ is decomposable
- c) $\text{Tor}_1^A(A/I, A/J)^\vee$ is not cyclic; here $(-)^\vee$ denotes the Matlis dual, and
- d) the mixed characteristic p represents a non-zero divisor on A/J ; in particular if J is a prime ideal.

Note that by Theorem (1.4) there exists an integer $t > 0$ such that (Q) has a positive answer for the pair $(A/I_{n,s}, A/J)$ for $s \geq t$.

The proof of Theorem (1.5) includes the reduction of MC on any local ring to MC on almost complete intersections. As a corollary to Theorem (1.7) we have the following:

Corollary. *Let C be an almost complete intersection local ring and let x_1, \dots, x_n be a system of parameters for C . Then x_1, \dots, x_n satisfies MC in the following cases:*

- i) $H_1(\underline{x}; C)$ is cyclic,
- ii) $H_1(\underline{x}; C)$ is decomposable,
- iii) $H_1(\underline{x}; C)^\vee$ is not cyclic, and

iv) p is not a non-zero divisor on C ; in particular when C is an almost complete intersection local domain.

The above results grew out of the first author's work in [7]. The question (Q), Theorem (1.4), parts b) and c) of Theorem (1.7) are generalizations and reformulations of results in [7] in the broader spectrum of regular local rings. This generality is done with the hope that the greater latitude of regular local rings together with the techniques for studying intersection multiplicities might throw more light into understanding this group of conjectures. The first author has been trying to prove part a) of Theorem (1.7) for a long time i.e., proving the validity of the monomial conjecture/canonical element conjecture when $H_1(\underline{x}; C)$, where $C = A/J$, (the first Koszul homology for a system of parameters $x_1 \dots x_n$ of C) is cyclic. Note that part b) of Theorem (1.7) says that it is valid when $H_1(\underline{x}; C)$ is decomposable. Finally, techniques of proving various aspects of the canonical element conjecture (see Theorem (1.6)) and a theorem of Kunz [20] came to the rescue. This result makes the first author hopeful about the prospect of the approach outlined in this section.

Section 2 is devoted to a re-examination of J. Koh's proof [19] of his significant result on validity of direct summand conjecture for degree p -extensions. Namely, he showed: if R is a regular local ring of mixed characteristic p such that R/pR is again regular, then a ring extension of the form $R \hookrightarrow R[\sqrt[p]{u}]'$ must be R -split; the notation $[\cdot]'$ refers to "integral closure". To support relevance of his result, Koh recalls from Hochster's article [15] that the direct summand conjecture reduces to the case of finite ring extensions $R \hookrightarrow B \hookrightarrow B[\sqrt[p]{u}]'$, where B is an intermediate normal domain. Thus Koh's main result addresses the special case $R = B$. Koh's argument requires a good deal of computation and somewhat tedious linear algebra. With the aid of "twenty-twenty" hindsight we propose here to give a more conceptual proof that relies on Galois theory and basic concepts of eigenvalues from linear algebra. After noting a simple criteria for a finite extension of normal domains $R \hookrightarrow A$ to be R -split (Proposition 2.1), we proceed to construct, within the context of Koh's hypothesis, a canonical free R -subalgebra S of A such that $pA \subseteq S$ (Theorem 2.2 and Theorem 2.3). The existence of such an S allows us to conclude $p \text{Ext}_R^1(A/R, R) = 0$. Since the short exact sequence $e : 0 \rightarrow R \rightarrow A \rightarrow A/R \rightarrow 0$ necessarily splits modulo p (the class $[e]$

represents an element of $p \operatorname{Ext}_R^1(A/R, R)$ we get that $R \hookrightarrow A$ is R -split.

In section 3 we return to the syzygy theorem of Evans-Griffith [10] and review its proof in light of the recent proof of the direct summand conjecture in dimension three by Heitmann [14]. Although Heitmann's result is for dimension 3, its impact by way of INIT allows us to argue that a non-free $(n - 2)$ nd syzygy of finite projective dimension over a local ring A of dimension n must have rank $\geq (n - 2)$ in any characteristic. Recall that the syzygy theorem of Evans-Griffith is valid in the equicharacteristic case. The above result leads to a proof of the syzygy conjecture for rings up to dimension ≤ 5 in mixed characteristic (Corollary 3.6). A graded version of the syzygy conjecture was shown to have an affirmative answer for mixed characteristic in [13].

This is not a joint work in the usual sense of the term. The first author's work is described in section 1 and the second author's work is described in sections 2 and 3. When the authors found out that they were writing up their articles on the same group of equivalent conjectures from different perspectives for the same issue of MMJ honoring Hochster on his 65th birthday, they decided to write it up as a single paper.

Before we get into the main body of our work we would like to state the syzygy conjecture and four equivalent versions of the canonical element conjecture which will be used in the next three sections of this article. Throughout this work, by a local ring we mean a noetherian local ring and subscript A (superscript A) will be omitted from the notations of Ext and Tor when there is no scope of ambiguity. For a module M over a ring A , $\dim M$ will denote its Krull dimension; for a local ring A , E will denote the injective hull of the residue field.

A. Syzygy Conjecture.

Let A be a local ring and let M be a finitely generated nonfree k th syzygy over A having finite projective dimension. Then $\operatorname{rank}_A M \geq k$.

B. Direct Summand Conjecture (DSC).

Let R be a regular local ring and let $i : R \hookrightarrow A$ be a module-finite extension of R . Then i splits as an R -module map.

C. Monomial Conjecture (MC).

Let A be a local ring of dimension n and x_1, \dots, x_n be a system of parameters of A . Then, for every integer $t > 0$,

$$(x_1 \dots x_n)^t \notin (x_1^{t+1}, \dots, x_n^{t+1}).$$

D. Canonical Element Conjecture (CEC).

Let A be a local ring of dimension n with maximal ideal m and residue field K . Let S_i denote the i th syzygy of K in a minimal resolution of K over A and let $\theta_n : \text{Ext}_A^n(K, S_n) \rightarrow H_m^n(S_n)$ denote the direct limit map. Then θ_n (class of Identity map on S_n) $\neq 0$.

E. Improved New Intersection Conjecture (INIC).

Let A be as above. Let F_\bullet be a complex of finitely generated free A -modules

$$F_\bullet : 0 \rightarrow F_s \rightarrow F_{s-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

such that $\ell(H_i(F_\bullet)) < \infty$ for $i > 0$ and $H_0(F_\bullet)$ has a minimal generator annihilated by a power of the maximal ideal m . Then $\dim A \leq s$.

SECTION 1

Let $(R, m, K = R/m)$ be a regular local ring and let M and N be two finitely generated R -modules such that $\ell(M \otimes_R N) < \infty$. Following Serre [26], we define $\chi^R(M, N) = \sum_{j \geq 0} (-1)^j \ell(\text{Tor}_j^R(M, N))$ and for $i > 0$, $\chi_i^R(M, N) = \sum_{j \geq 0} (-1)^j \ell(\text{Tor}_{i+j}^R(M, N))$. We drop R from the notations when there is no scope of ambiguity. Note that the above definitions make sense over any local ring A provided at least one of the modules has finite projective dimension. In [15] Hochster proved that for the validity of the direct summand conjecture it is enough to assume the regular local ring to be unramified.

Now we will prove our first proposition. This will involve several results from Serre [26] and Lichtenbaum [21] on intersection multiplicities.

Proposition 1.1. *Let R be an equicharacteristic or unramified regular local ring and let M, N be two finitely generated R -modules such that $\ell(M \otimes_R N) < \infty$. We have the*

following:

- a) if $\dim M + \dim N < \dim R$ and $\text{depth } M + \text{depth } N = \dim R - 1$, then $\ell(M \otimes_R N) = \ell(\text{Tor}_1^R(M, N))$;
- b) if $\dim M + \dim N < \dim R$ and $\text{depth } M + \text{depth } N < \dim R - 1$, then (Q) has a negative answer;
- c) if $\dim M + \dim N = \dim R$ and $\text{depth } M + \text{depth } N = \dim R - 1$, then (Q) has a positive answer.

Proof. We have, from the above definitions, $\ell(M \otimes_R N) - \ell(\text{Tor}_1^R(M, N)) = \chi(M, N) - \chi_2(M, N)$.

Now we would like to recall the following results on equicharacteristic/unramified regular local rings when $\ell(M \otimes_R N) < \infty$:

- i) $\dim M + \dim N \leq \dim R$ (Th. 3, Ch. V, [26]) (true for any regular local ring);
- ii) $\chi(M, N) \geq 0$, the sign of equality holds if and only if $\dim M + \dim N < \dim R$ (Th. 1, Lemma, Ch. V, [26]);
- iii) $\chi_i(M, N) \geq 0$, the sign of equality holds if and only if $\text{Tor}_j(M, N) = 0$ for $j \geq i$ ([21], [26]); and
- iv) if $i = \dim R - \text{depth } M - \text{depth } N$, then $\text{Tor}_j(M, N) = 0$ for $j > i$ (Th. 4, Ch. V, [26]) (true in more generality).

The proof now follows directly from the above observation.

- a) if $\dim M + \dim N < \dim R$ and $\text{depth } M + \text{depth } N = \dim R - 1$, then by ii) and iv), we have $0 = \chi(M, N) = \ell(M \otimes_R N) - \ell(\text{Tor}_1(M, N))$;
- b) if $\dim M + \dim N < \dim R$ and $\text{depth } M + \text{depth } N < \dim R - 1$, then, by ii) and iii) and iv), we have $0 = \chi(M, N) = \ell(M \otimes N) - \ell(\text{Tor}_1(M, N)) + \chi_2(M, N)$ and $\chi_2(M, N) > 0$;
- c) if $\dim M + \dim N = \dim R$ and $\text{depth } M + \text{depth } N = \dim R - 1$, then, by ii) and iv), we have $0 < \chi(M, N) = \ell(M \otimes N) - \ell(\text{Tor}_1(M, N))$.

Remarks.

1. The answer to (Q) is clear when $\dim M + \dim N < \dim R$ i.e., when vanishing holds. For the positivity case i.e., when $\dim M + \dim R = \dim R$, it will be highly significant to

understand the situation when $\dim R - \text{depth } M - \text{depth } N = 2$. In this case, by the result mentioned above, $\chi(M, N) = \ell(M \otimes N) - \ell(\text{Tor}_1(M, N)) + \ell(\text{Tor}_2(M, N))$. Recall that it was observed in [9] that the general question on positivity can be reduced to the above situation.

2. The statement in c) is valid over any Cohen-Macaulay local ring for a pair of modules one of which is perfect and the other has $(\dim - \text{depth}) = 1$.

3. Serre observed in [26] that in the equicharacteristic case, for the positivity part, $\chi(M, N) \geq e_m(M)e_m(N)$, where $e_m(T)$ denotes the Hilbert-Samuel multiplicity of a finitely generated R -module T . This makes us raise the following very difficult question: Is $\chi_2(M, N) < e_m(M)e_m(N)$? We do not know the answer even in the best possible non-obvious situation.

Recall that both Serre's work on intersection multiplicities [26] and Lichtenbaum's work on the χ_i problem [21] depended heavily on the following observation: Let (A, m, K) be a local ring and let M be a finitely generated A -module. Let I be an ideal of A generated by an A -sequence such that $\ell(M/IM) < \infty$. Then i) $\chi(A/I, M) \geq 0$, equality holds if and only if $\dim M < ht I$ and ii) $\chi_i(A/I, M) \geq 0$, equality holds if and only if $\text{Tor}_j(A/I, M) = 0$ for $j \geq i$. However, even when A is regular, I is as above and $\dim M + \dim A/I = \dim A$, we do not know, in general, the answer to (Q) for the pair $(A/I, M)$. The best result we can prove in this situation is stated in Theorem (1.4). For our proof of this theorem first we define superficial elements for a finitely generated module over a local ring A by extending the original definition of Samuel [25] for local rings.

Definition. Let (A, m, K) be a local ring, M be a finitely generated A -module and I be an ideal of A . An element $x \in I^s$ is called a superficial element of order s for M in I , if there exists an integer a such that $(I^n M : x) \cap I^a M = I^{n-s} M$ whenever $n \geq s + a$.

The existence and properties of superficial elements are described in the following Lemma.

Lemma 1.2. *A, I, M as above. We have the following:*

- i) *There exists a superficial element $x \in I$ for M of some order s .*

- ii) If $\text{depth}_I M > 0$, x can be chosen to be a non-zero divisor.
- iii) Assume that $K = A/m$ is infinite. Then there exists superficial elements for M of any given order. Henceforth we assume that K is infinite.
- iv) Assume that $\ell(M/IM) < \infty$. If x is a superficial element of order s , then $e(I; M/xM) = s e(I; M)$. Here $e(I; M) =$ the Hilbert-Samuel multiplicity of M with respect to the ideal I . And
- v) If I is minimally generated by a system of parameters of M , one can choose x_1, \dots, x_n in I such that a) $I = (x_1, \dots, x_n)$, b) x_1, \dots, x_{n-1} are superficial elements of M and c) $e(I; M) = e((x_2, \dots, x_n), M/x_1M) = \dots = e(x_n, M/(x_1, \dots, x_{n-1})M)$.

Proof. The proof of these facts are essentially the same as outlined by Samuel for local rings in [25]. We leave this as an exercise to the reader.

Now we want to prove the following.

Proposition 1.3. *Let (A, m, K) be a local ring, M be a finitely generated A -module and let x_1, \dots, x_n be an A -sequence such that it is a system of parameters for M . Assume that $e((x_1, \dots, x_n); M) = e((x_2, \dots, x_n); M/x_1M) = \dots = e(x_n; M/(x_1, \dots, x_{n-1})M)$. Then $\ell(H_i(x_1, \dots, x_{n-1}; M)) < \infty$ for $i > 0$.*

Proof. We write \underline{x} for the ideal (x_1, \dots, x_n) and \underline{x}_{n-1} for the ideal (x_1, \dots, x_{n-1}) . Let P_1, \dots, P_r denote the minimal primes in $\text{Ass}_A(M/\underline{x}_{n-1}M)$. Then

$$e(x_n; M/\underline{x}_{n-1}M) = \sum_{i=1}^r \ell((M/x_{n-1}M)_{P_i}) e(x_n; A/P_i). \quad (1)$$

On the other hand by the associativity formula for Hilbert-Samuel multiplicity, we have

$$e(\underline{x}; M) = \sum_{i=1}^r e(x_n; A/P_i) e(\underline{x}_{n-1}; M_{P_i}) \quad (2)$$

(we refer the reader to [22], [25] for (1) and (2)).

Recall that

$$e(\underline{x}_{n-1}; M_{P_i}) = \sum_{j=0}^{n-1} (-1)^j \ell(H_j(\underline{x}_{n-1}; M_{P_i})) \quad (3)$$

(see [25]).

By the result mentioned in the beginning of 1.2, we know that $\chi_1^{(i)} = \sum_{j=0}^{n-2} (-1)^j \ell(H_{j+1}(\underline{x}_{n-1}, M_{P_i})) \geq 0$ and $= 0$ if and only if $H_j(\underline{x}_{n-1}; M_{P_i}) = 0$ for $j \geq 1$ ([21]). Now subtracting (2) from (1), due to our assumption, we obtain

$$0 = \sum_{i=1}^r e(x_n; A/P_i) \chi_1^{(i)}.$$

Since $e(x_n; A/P_i) > 0$, we must have $\chi_1^{(i)} = 0$ for $i = 1, 2, \dots, r$. This implies, by the result stated in the beginning of (1.2), that $H_j(x_{n-1}; M_{P_i}) = 0$ for every $j \geq 1$ and for $i = 1, \dots, r$. Hence $\ell(H_j(x_{n-1}, M)) < \infty$ for every $j \geq 1$.

Now we are ready to prove our next theorem.

Theorem 1.4. *Let (A, m, K) be a local ring and let M be a finitely generated A -module. Let I be an ideal of A generated by an A -sequence of length n such that $\ell(M/IM) < \infty$ and $\dim A/I + \dim M = \dim A$. Then there exists an integer $t > 0$ and a minimal set of generators x_1, \dots, x_n of I such that (Q) has a positive answer for pairs $(A/I_{n,s}, M)$ where $I_{n,s} = (x_1, \dots, x_{n-1}, x_n^s)$, for $s \geq t$.*

Proof. Let $I = (y_1, \dots, y_n)$ where y_1, \dots, y_n form an A -sequence. By assumption y_1, \dots, y_n form a system of parameters for M . Due to part v) of Lemma (1.2) we can construct $x_1, \dots, x_{n-1} \in I$ such that a) $I = (x_1, \dots, x_{n-1}, y_n)$ and b) $e(I, M) = e((x_2, \dots, x_{n-1}, y_n), M/x_1M) = \dots = e(y_n, M/(x_1, \dots, x_{n-1})M)$. Write $\underline{x} = (x_1, \dots, x_{n-1})$. Note that $\text{Tor}_i^A(A/I, M) = H_i(\underline{x}, y_n; M)$ for $i \geq 0$. We have a short exact sequence

$$0 \rightarrow \frac{H_1(\underline{x}; M)}{y_n^t H_1(\underline{x}; M)} \rightarrow H_1(\underline{x}, y_n^t; M) \rightarrow (o : y_n^t)_{M/\underline{x}M} \rightarrow 0.$$

By the previous proposition, $\ell(H_j(\underline{x}; M)) < \infty$ for $j > 0$. Then, for $t \gg 0$, $y_n^t H_1(\underline{x}; M) = 0$ and $\ell((o : y_n^t)_{M/\underline{x}M})$ is constant. However, $\ell(M/(\underline{x}, y_n^t)M)$ is a strictly increasing function of t . Hence $\ell(M/(\underline{x}, y_n^t)M) > \ell(H_1(\underline{x}, y_n^t; M)) = \ell(\text{Tor}_1^A(M, A/(\underline{x}, y_n^t)))$ for $t \gg 0$ and the proof of our theorem is complete.

Our next theorem demonstrates why a definite answer to (Q), even when one of the modules is a complete intersection, is so difficult to comprehend. This reveals the relation

between the MC and a special case of a question on intersection multiplicity in terms of χ and χ_2 .

Theorem 1.5. *MC is valid for all local rings A if and only if for every unramified/equicharacteristic regular local ring R and for every pair of ideals I, J of R such that i) I is a complete intersection ideal, ii) J is an almost complete intersection ideal (i.e., minimally generated by $(ht J + 1)$ elements), iii) $ht I + ht J = \dim R$ and iv) $I + J$ is primary to the maximal ideal of R , $\ell(R/(I + J)) > \ell(\text{Tor}_1^R(R/I, R/J))$.*

Proof. The proof will be completed in several steps. Note that for monomial conjecture (or for any other equivalent form) we can always assume A is a complete local normal domain. Hence we write $A = R/\tilde{P}$, where R is an unramified or equicharacteristic complete regular local ring. Let $S = R/\underline{\xi}$, where $\underline{\xi}$ is the ideal generated by a maximal R -sequence ξ_1, \dots, ξ_r contained in P . Then $A = S/P$, where $P = \tilde{P}/\underline{\xi}$. Write $\Omega = \text{Hom}_S(A, S)$ —the canonical module for A ; Ω is an ideal of S . Let E denote the injective hull of the residue field of S .

Step 1. Here we sketch a short proof of the following theorem due to Strooker and Stückrad on a characterization of MC (the first author independently proved a similar characterization for DSC [7]).

Theorem ([27]). *(With notations as above.) A satisfies MC if and only if for every system of parameters x_1, \dots, x_n of S , $\Omega \not\subset (x_1, \dots, x_n)$.*

Proof. Let y_1, \dots, y_n be a system of parameters of A . We can lift it to a system of parameters x_1, \dots, x_n for S such that $\text{im}(x_i) = y_i$, $1 \leq i \leq n$. Conversely any system of parameters for S is a system of parameters for A . Write $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$. MC for A is equivalent to the assertion that for every system of parameters y_1, \dots, y_n of A the direct limit map $\alpha : A/\underline{y} \rightarrow H_m^n(A)$ is non-null [15]. Since S is a complete intersection, the direct limit map $\beta : S/\underline{x} \rightarrow H_{m_S}^n(S)$ is non-null. We write

$T^\vee = \text{Hom}_S(T, E)$ for any S -module T . We have the following commutative diagram:

$$\begin{array}{ccc}
S/\underline{x} & \xrightarrow{\beta} & H_{m_s}^n(S) \\
& \searrow & \\
\downarrow \eta & & \downarrow H_{m_s}^n(\eta) = \gamma \\
A/\underline{y} & \xrightarrow{\alpha} & H_m^n(A)
\end{array}$$

where η denotes the natural surjection. This implies that α is non-null $\Leftrightarrow \alpha_0\eta = \gamma_0\beta$ is non-null $\Leftrightarrow H_m^n(A)^\vee \rightarrow (S/\underline{x})^\vee$ is non-null $\Leftrightarrow \text{Im}(\Omega \rightarrow S/\underline{x})$ is non-null $\Leftrightarrow \Omega \not\subseteq \underline{x}$ (Recall, by local duality $H_m^n(A)^\vee = \Omega$).

Step 2. In this step we reduce MC for all local rings to MC for local almost complete intersections. We prove the following:

MC is valid for all local rings if and only if MC holds for all local almost complete intersections.

Proof. Suppose MC holds for all local almost complete intersections. Let A be a complete local domain. Then we have $A = R/\tilde{P}$, where R is a complete regular local ring. Since $R_{\tilde{P}}$ is a regular local ring, one can choose a maximal R -sequence ξ_1, \dots, ξ_r in \tilde{P} such that $\tilde{P}R_{\tilde{P}} = (\xi_1, \dots, \xi_r)R_{\tilde{P}}$. Write $S = R/\underline{\xi}$, where $\underline{\xi} = (\xi_1, \dots, \xi_r)$ and $P = \tilde{P}/\underline{\xi}$. Then S is a complete intersection, $A = S/P$, $\dim S = \dim A$ and $PS_P = 0$. Let $\Omega = \text{Hom}_S(A, S)$ —the canonical module of S . Consider the primary decomposition in S : $0 = P \cap q_2 \cap \dots \cap q_h$, where q_i is P_i -primary and $ht P_i = ht P = 0$, $2 \leq i \leq h$. It can be checked easily that $\Omega = q_2 \cap \dots \cap q_h$. Choose $\lambda \in P - \bigcup_{i \geq 2} P_i$. Then $\Omega = \text{Hom}(S/\lambda S, S)$ and $S/\lambda S$ is an almost complete intersection. Since, by assumption, $S/\lambda S$ satisfies MC, by step 1, $\Omega \not\subseteq$ the ideal generated by any system of parameters in S . Hence, again by step 1, A satisfies MC.

Step 3. Now we prove the assertion in our theorem. First assume that every pair (I, J) , as mentioned in the statement of the theorem, satisfies the length inequality i.e., $\ell(R/(I + J)) > \ell(\text{Tor}_1^R(R/I, R/J))$. By step 2, we can assume A is an almost complete intersection ring of the form $S/\lambda S$, where S is a complete intersection, and $\dim S = \dim A$. Write $\Omega = \text{Hom}(S/\lambda S, S)$ —the canonical module for A . Consider the short exact sequence

$$0 \rightarrow S/\Omega \xrightarrow{f} S \rightarrow S/\lambda S \rightarrow 0, \quad (1)$$

where $f(\bar{1}) = \lambda$. Let x''_1, \dots, x''_n be a system of parameters for A . We can lift x''_1, \dots, x''_n to x'_1, \dots, x'_n in S in such a way that $\{x'_1, \dots, x'_n\}$ form a system of parameters in S . Write $\underline{x}'' = (x''_1, \dots, x''_n)$ and $\underline{x}' = (x'_1, \dots, x'_n)$. Tensoring (1) with S/\underline{x}' we obtain an exact sequence

$$0 \rightarrow \mathrm{Tor}_1^S(S/\underline{x}', S/\lambda S) \rightarrow S/(\Omega + \underline{x}') \xrightarrow{\bar{f}} S/\underline{x}' \rightarrow S/(\underline{x}' + \lambda S) \rightarrow 0 \quad (2)$$

where \bar{f} is induced by f and $\mathrm{Tor}_1^S(S/\underline{x}', S/\lambda S) = H_1(\underline{x}'; S/\lambda S) = H_1(\underline{x}'; A)$.

Then

$$\Omega \not\subset \underline{x}' \Leftrightarrow \ell(S/(\underline{x}' + \lambda S)) > \ell(\mathrm{Tor}_1(S/\underline{x}', S/\lambda S)). \quad (3)$$

Now lift x'_1, \dots, x'_n to an R -sequence x_1, \dots, x_n in R . Write $I = \underline{x}$ and $J = (\xi, \lambda)$. Then the above condition translates to $\ell(R/(I + J)) > \ell(\mathrm{Tor}_1(R/I, R/J))$ —as required in our statement.

For the converse part of our theorem, write $I = (x_1, \dots, x_n)$, $J = (y_1, \dots, y_r, \lambda)$ where $x_1, \dots, x_n, y_1, \dots, y_r$ form R -sequences such that $n + r = \dim R$. Let $S = R/(y_1, \dots, y_r)$, $A = S/\lambda S$ and let $x'_i = \mathrm{im}(x_i)$ in S , $1 \leq i \leq n$. Write $\underline{x}' = (x'_1, \dots, x'_n)$ and $\underline{x} = (x_1, \dots, x_n)$. Since $I + J$ is primary to the maximal ideal and both R/\underline{x} and S are complete intersections, by a result of Serre mentioned earlier, $\mathrm{Tor}_i^R(R/\underline{x}, S) = 0$ for $i > 0$. This implies that $\mathrm{Tor}_i^R(R/I, R/J) = \mathrm{Tor}_i^S(S/\underline{x}', A)$, $i \geq 0$. Let $\Omega = \mathrm{Hom}_S(A, S)$ —the canonical module for A . Now (1), (2), (3) and the subsequent arguments complete the proof.

As a corollary we derive the following:

Corollary 1. *With notations as in the theorem, $\ell(R/(I + J)) \geq \ell(\mathrm{Tor}_1^R(R/I, R/J))$.*

The proof follows from exact sequence (2) in the proof of the theorem above.

Corollary 2. *Let A be a complete local domain; x_1, \dots, x_n a system of parameters for A . Then there exists $y_1, \dots, y_{n-1} \in (x_1, \dots, x_n)$ such that $(y_1, \dots, y_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n)$ and $(y_1, \dots, y_{n-1}, x_n^t)$ satisfies MC for $t \gg 0$.*

The proof follows from Theorem (1.4) and the equivalence (3) in the proof of Theorem (1.5).

As mentioned in the introduction, for the past several years we have been trying to prove that MC/CEC holds for a system of parameters x_1, \dots, x_n of local ring A , if $H_1(\underline{x}; A)$ is cyclic. We could prove the conjecture if $H_1(\underline{x}; A)$ is decomposable, but a proof for first important case of indecomposability, i.e., the cyclic case, always eluded us. Finally we are now able to prove this case when A is an almost complete intersection ring. The following theorem plays a crucial role in the proof of this result. Our proof of this theorem will involve several results from [4] and [6] where the first author studied various aspects of the canonical element conjecture.

Theorem 1.6. *Let R be a regular local ring and I and J be two ideals of R satisfying the following conditions: i) I is a complete intersection, ii) J is an almost complete intersection, iii) $ht I + ht J = \dim R$ and iv) $I + J$ is m -primary. Then for the validity of MC over all local rings it is enough to consider the case where $I + J$ is not a complete intersection ideal in R .*

Proof. We will complete the proof in the following steps.

Step 1. Let (A, m, K) be a local ring of dimension n . Let $S_i = i$ th syzygy of K in a minimal resolution of K over A . Let $\theta_i : \text{Ext}^i(K, S_i) \rightarrow H_m^i(S_i)$ denote the direct limit map and let $\eta_i = \theta_i$ (Image of Identity map on S_i). CEC demands that $\eta_n \neq 0$ [17].

Now consider a minimal resolution $F_\bullet = \{A^{t_i}, d_i\}_{i \geq 0}$ of K and break it up into short exact sequences:

$$0 \rightarrow S_n \rightarrow A^{t_{n-1}} \rightarrow S_{n-1} \rightarrow 0, 0 \rightarrow S_{n-1} \rightarrow A^{t_{n-2}} \rightarrow S_{n-2} \rightarrow 0 \cdots 0 \rightarrow S_1 \rightarrow A \rightarrow K \rightarrow 0. \quad (1)$$

These sequences give rise to the following commutative diagram:

$$\begin{array}{ccccccccc} K & \xrightarrow{\delta_0} & \text{Ext}_A^1(K, S_1) & \xrightarrow{\delta_1} & \text{Ext}_A^2(K, S_2) & \xrightarrow{\delta_2} & \cdots & \longrightarrow & \text{Ext}_A^{n-1}(K, S_{n-1}) & \xrightarrow{\delta_{n-1}} & \text{Ext}_A^n(K, S_n) \\ \parallel & & \downarrow \theta_1 & & \downarrow \theta_2 & & & & \downarrow \theta_{n-1} & & \downarrow \theta_n \\ H_m^0(K) & \xrightarrow{\tilde{\delta}_0} & H_m^1(S_1) & \xrightarrow{\tilde{\delta}_1} & H_m^2(S_2) & \xrightarrow{\tilde{\delta}_2} & \cdots & \longrightarrow & H_m^{n-1}(S_{n-1}) & \xrightarrow{\tilde{\delta}_{n-1}} & H_m^n(S_n) \end{array} \quad (2)$$

where all the horizontal maps are connecting homomorphisms obtained from the short exact sequences above. It follows from above that $\tilde{\delta}_i(\eta_i) = \eta_{i+1}$ for $0 \leq i < n$. Hence η_n

is nothing but the image of $1 \in K$ at the upper left hand corner. We have the following theorem:

Theorem ([4]). *With notations as above, $\eta_i \neq 0$ for $0 \leq i \leq n - 1$.*

Since the techniques involved in this proof are completely different from those we are dealing with in this paper, we refer the reader to [4] for a proof.

Corollary. *Let A be a local ring with notations as above. Suppose that $H_m^{n-1}(A) = 0$. Then CEC holds for A .*

Proof. Since $H_m^{n-1}(A) = 0$, from the first short exact sequence in (1), we have a short exact sequence: $0 \rightarrow H_m^{n-1}(S_{n-1}) \rightarrow H_m^n(S_n)$. Now, by the above theorem, the proof is complete.

Before our next step we would like to recall that for the validity of CEC etc. we can assume, without any loss of generality, that the given local ring A is a complete local normal domain ([17]).

Step 2. Let A be a complete local normal domain. Then, as described earlier, we can write $A = S/P$, S a complete intersection such that $\dim S = \dim A$. Write $\Omega = \text{Hom}_S(A, S)$ —the canonical module for A . Then S/Ω satisfies CEC.

Proof. (This was proved in [6] using dualizing complexes.)

Since A is normal domain, $\text{Hom}_S(\Omega, S) = \text{Hom}_S(\Omega, \Omega) = \text{Hom}_A(\Omega, \Omega) \simeq A$. We can construct S in such a way that (Step 2 of proof of Theorem (1.5)) $\text{Hom}_S(S/\Omega, S) = P$. We consider the following short exact sequence

$$0 \rightarrow \Omega \rightarrow S \rightarrow S/\Omega \rightarrow 0.$$

Applying $\text{Hom}_S(-, S)$ to this short exact sequence we obtain the following short exact sequence

$$0 \rightarrow P \hookrightarrow S \rightarrow A \rightarrow \text{Ext}_S^1(S/\Omega, S) \rightarrow 0.$$

Since $A = S/P$, this implies that $\text{Ext}_S^1(S/\Omega, S) = 0$. This implies, by local duality, $H_m^{n-1}(S/\Omega) = 0$. Hence we are done by the corollary in step 1.

Step 3. In this step we show that given a system of parameters x_1, \dots, x_n of S , we can choose $\lambda \in P$ in such a way that $\lambda \notin (x_1, \dots, x_n)$ (Step 2, Th. (1.5)).

As described earlier (Step 2, Th. (1.5)), we can choose S in such a way that $PS_P = 0$. Since S/Ω satisfies CEC, by step 1 of Theorem 1.5 we conclude that $P = \text{Hom}_S(S/\Omega, S)$ —the canonical module for S/Ω , is not contained in the ideal generated by any system of parameters of S . Thus given x_1, \dots, x_n , a system of parameters of S , we can choose $\lambda \in P - [(x_1, \dots, x_n) \cup (\cup P_i, i \geq 2)]$, where P_i s are as in step 2 in the proof of Theorem (1.5).

Now, by the last part of the proof of Theorem (1.5), the proof of this theorem is complete.

Now we are ready to prove our final theorem of this section. With notations as in Theorem (1.5), we can assume due to Theorem (1.6) that $(I + J)$ is not a complete intersection ideal. This assumption will be used only for the proof of part a) of our theorem—for other parts no such assumption is necessary.

Theorem 1.7. *Let R, I, J be as in Theorem (1.5) satisfying conditions i), ii), iii) and iv). Then $\ell(R/(I + J)) > \ell(\text{Tor}_1(R/I, R/J))$ in the following cases:*

- a) $\text{Tor}_1(R/I, R/J)$ is cyclic,
- b) $\text{Tor}_1(R/I, R/J)$ is decomposable,
- c) $[\text{Tor}_1(R/I, R/J)]^\vee$ is not cyclic ($_^\vee = \text{Hom}(_, E)$), and
- d) the mixed characteristic p is not a zero-divisor on R/J , in particular J is a prime ideal.

Proof. Let $I = (x_1, \dots, x_n)$, $J = (y_1, \dots, y_r, \lambda)$, $\underline{y} = (y_1, \dots, y_r)$, $S = R/\underline{y}$ and $A = R/J$. Then $\Omega = \text{Hom}_S(S/\lambda S, S) = \text{Hom}_S(A, S)$. Write $\xi_i = \text{im}(x_i)$ in S ; then ξ_1, \dots, ξ_n is a system of parameters for the complete intersection S . Write $\underline{\xi} = (\xi_1, \dots, \xi_n)$.

a) By Theorem (1.6) we can assume $I + J$ is not a complete intersection ideal in R .

Consider the exact sequence

$$0 \rightarrow S/\Omega \xrightarrow{f} S \rightarrow S/\lambda S \rightarrow 0.$$

Tensoring with R/I , we obtain an exact sequence

$$0 \rightarrow \mathrm{Tor}_1^R(R/I, R/J) \rightarrow S/(\Omega + \underline{\xi}) \xrightarrow{\bar{f}} S/\underline{\xi} \rightarrow S/(\underline{\xi} + \lambda S) \rightarrow 0.$$

If MC does not hold on $S/\lambda S$ with respect to the system of parameters ξ_1, \dots, ξ_n , then $\Omega \subset (\xi_1, \dots, \xi_n)$ and \bar{f} boils down to multiplication by λ on $S/\underline{\xi}$. Hence $H_1(\underline{\xi}; S/\lambda S) = \mathrm{Tor}_1^R(R/I, R/J) = (0 : \lambda)S/\underline{\xi} =$ the injective hull of the residue field K over the ring $S/(\underline{\xi} + \lambda S)$. Now consider the 0-dimensional complete intersection ring $B = R/(I + \underline{y}) = S/\underline{\xi}$. Write $\mu = \mathrm{Im}(\lambda)$ in B . Then $\Omega_{B/\mu B} = (0 : \mu)B = (0 : \lambda)S/\underline{\xi}$; $B/\mu B = S/(\underline{\xi} + \lambda S)$. By assumption $\mathrm{Tor}_1^R(R/I, R/J)$ is cyclic; hence $(0 : \lambda)S/\underline{\xi}$ is cyclic. Since $\ell((0 : \lambda)S/\underline{\xi}) = \ell(S/(\underline{\xi} + \lambda S))$, this implies that $(0 : \lambda)S/\underline{\xi} \simeq S/(\underline{\xi} + \lambda S)$. Thus $\Omega_{B/\mu B} \simeq B/\mu B$ i.e., $B/\mu B$ is Gorenstein. By a theorem of Kunz [21], this implies that $B/\mu B = S/(\underline{\xi} + \lambda S)$ is a complete intersection. This contradicts the fact that $I + J$ is not a complete intersection.

b) As pointed out in part a), if MC fails, $\mathrm{Tor}_1(R/I, R/J) = H_1(\underline{\xi}; A) = (0 : \lambda)S/\underline{\xi}$ —the injective hull of K over the local ring $S/(\underline{\xi} + \lambda S)$. Hence $\mathrm{Tor}_1(R/I, R/J)$ is indecomposable.

c) If MC fails, $[\mathrm{Tor}_1^R(R/I, R/J)]^\vee = H_1(\underline{\xi}; A)^\vee = [(0 : \lambda)S/\underline{\xi}]^\vee$ is a cyclic module, since $(0 : \lambda)S/\underline{\xi}$ is the injective hull of K over the 0-dimensional local ring $S/(\underline{\xi} + \lambda S)$.

d) We need the following Lemma and the theorem on CEC thereafter to prove our assertion.

Lemma. *Let A be an almost complete intersection ring i.e., $A = S/\lambda S$ as above, S is a complete intersection and $\dim S = \dim A$. Let x be a non-zero divisor on S and A . Then x is a non-zero divisor on $\mathrm{Ext}_S^1(A, S)$.*

Proof. Consider the short exact sequence

$$\theta \rightarrow S \xrightarrow{x} S \xrightarrow{\eta} S/xS \rightarrow 0.$$

Applying $\mathrm{Hom}_S(A, _)$ we obtain the following exact sequence

$$0 \rightarrow \frac{\Omega}{x\Omega} \xrightarrow{\bar{\eta}} \Omega_{A/xA} \rightarrow \mathrm{Ext}_S^1(A, S) \xrightarrow{x} \mathrm{Ext}_S^1(A, S) \rightarrow .$$

Here $\Omega = \text{Hom}_S(A, S)$, $\Omega_{A/xA} = \text{Hom}_S(A, S/xS) = \text{Hom}_S(A/xA, S/xS)$.

Claim. The map $\bar{\eta}$ is onto.

Let $y \in S$ be such that $\lambda y \in xS$; write $\lambda y = x\mu$. Since x is a non-zero divisor on $A = S/\lambda S$, $\mu = \lambda b$, $b \in S$. Hence $\lambda y = x\mu = x\lambda b$ i.e., $\lambda(y - xb) = 0$. Thus $y - xb \in \Omega$. Hence the claim.

The proof of the Lemma now follows from the above exact sequence.

Theorem. *Let A be a local ring of the form S/I , S a complete intersection, such that $\dim S = \dim A$. Let x be a non-zero divisor on A and $\text{Ext}_S^1(A, S)$. Then A satisfies CEC if and only if A/xA does so.*

The “if” part is due to Hochster [15] and the “only if” part is due to the first author [5]. Since the complete proof uses dualizing complex we refrain from giving the proof here; instead we refer the reader to [5].

Since A/pA satisfies CEC ([15]) our proof follows from the above Lemma and the theorem. Hence proof of Theorem (1.7) is complete.

Corollary. *Let A be an almost complete intersection ring and let x_1, \dots, x_n be a system of parameters for A . Then x_1, \dots, x_n satisfies MC in the following cases:*

- i) $H_1(\underline{x}; A)$ is cyclic,
- ii) $H_1(\underline{x}; A)$ is decomposable,
- iii) $H_1(\underline{x}; A)^\vee$ is not cyclic, and
- iv) p is not a zero-divisor on A ; in particular A is an almost complete intersection domain.

The proof is immediate from the above theorem. Recall that we reduced the validity of MC over all local rings to its validity on almost complete intersection rings in the proof of Theorem (1.5) ([7]).

SECTION 2

As noted in the introduction Koh’s result [19] provides an affirmative answer to the

direct summand conjecture for the case $R \hookrightarrow A$ when A represents the integral closure of a p th root extension of R . We begin with a general observation that allows one to conclude that a finite ring extension $A \hookrightarrow B$ of normal domains is A -split, i.e., the short exact sequence $e : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is A -split exact where $C = B/A$.

Proposition 2.1. *Suppose $A \hookrightarrow B$ is a finite extension of integral domains for which A is local and integrally closed. Let $x \in m_A - \{0\}$ and consider the short exact sequence $e : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and its class $[e] \in \text{Ext}_A^1(C, A)$.*

- i) *If $A/xA \rightarrow B/xB$ is A -split, or equivalently, A/xA -split, then $[e] \in x \text{Ext}_A^1(C, A)$.*
- ii) *If B contains an A -free submodule F such that $A \subseteq F \subseteq B$ and such that $x B \subseteq F$, then $x \text{Ext}_A^1(C, \bullet) \equiv 0$.*
- iii) *If the hypotheses of both (i) and (ii) hold simultaneously then $A \hookrightarrow B$ is A -split.*

Proof. (i) Since A is integrally closed and B is an integral domain it follows that $C = B/A$ is necessarily a torsion-free A -module (that is, a relation $xb = a$ means that b is in the fraction field of A and hence $b \in A$). So the element $x \in m_A - \{0\}$ is necessarily regular on C from which it follows that the induced map $A/xA \rightarrow B/xB$ is an injective ring homomorphism. In addition, the short exact sequence $0 \rightarrow A \xrightarrow{x} A \rightarrow \overline{A} \rightarrow 0$ induces a “change of rings” long exact sequence (see the discussion in [12, p. 5]) on cohomology

$$\begin{aligned} 0 \rightarrow \text{Hom}_A(C, A) \xrightarrow{x} \text{Hom}_A(C, A) \rightarrow \text{Hom}_{\overline{A}}(\overline{C}, \overline{A}) \xrightarrow{\delta} \\ \text{Ext}_A^1(C, A) \xrightarrow{x} \text{Ext}_A^1(C, A) \rightarrow \text{Ext}_{\overline{A}}^1(\overline{C}, \overline{A}) \rightarrow \dots \end{aligned}$$

where $\overline{(\cdot)}$ indicates “modulo x ”. Under the assumption in part (i), we have that the class $[e]$ is sent to zero in $\text{Ext}_{\overline{A}}^1(\overline{C}, \overline{A})$. It readily follows that $[e] \in x \text{Ext}_A^1(C, A)$.

(ii) First we observe that $A \subseteq F$ is necessarily A -split since “1” cannot lie in $m_A F$; so $G = F/A$ is an A -free submodule of C . Moreover, our hypothesis in (ii) guarantees that $x C \subseteq G \subseteq C$. A standard argument in elementary homological algebra shows $x \text{Ext}_A^1(C, \bullet) \equiv 0$.

(iii) The claim here is a trivial consequence of parts (i) and (ii) since $[e] \in x \text{Ext}_A^1(C, A)$ while $x \text{Ext}_A^1(C, A) = 0$.

In order to apply the preceding criteria in our proof of Koh's theorem we must first set up some notation and a construction. Since Koh reduces his argument quickly to the case where R is a complete local ring, we assume that V is a complete dvr of mixed characteristic p in which p generates the maximal ideal in V . We set $R = V[[X_1, \dots, X_n]]$ and consider a finite extension $R \hookrightarrow A$ where $A = R[\sqrt[p]{u}]'$. We intend to construct a free R -algebra S in A such that $pA \subseteq S$. To this end let ζ be a primitive p^{th} root of unity in a field extension of the fraction field of V and let V' represent the integral closure of $V[\zeta]$. So V' is a complete dvr. Now ζ is a root of the polynomial $f(X) = 1 + X + X^2 + \dots + X^{p-1} = (X^p - 1)/(X - 1)$. Since p is a prime element in V one gets that $f(X)$ is irreducible in $V[X]$ by noticing that $f(X + 1) = [(X + 1)^p - 1]/X$ is irreducible from Eisenstein's criteria. Therefore $[V' : V] = p - 1$. We set $R' = V'[[X_1, \dots, X_n]]$ and observe that R' is a complete regular local ring such that R' is R -free of rank $p - 1$. From the associated diagram of fraction fields,

$$\begin{array}{ccc} L & \xrightarrow{p-1} & L' = K[\sqrt[p]{u}, \zeta] \\ p \Big| & & p \Big| \\ K & \xrightarrow{p-1} & K' = K[\zeta] \end{array}$$

where K and L are the fraction fields of R and A , respectively, one sees that L'/K and L'/K' are Galois extensions. In particular, L'/K' is a Kummer extension. From the commutative diagram of fraction fields we get a corresponding diagram of finite ring extensions

$$\begin{array}{ccc} A & \xrightarrow{p-1} & A' \\ p \Big| & & p \Big| \\ R & \xrightarrow{p-1} & R' \end{array}$$

where R and R' are regular local rings and A' is the integral closure of R in L' . We denote the Galois group of L'/K by G and the corresponding (necessarily) normal subgroup $H = \text{Gal}(L'/K')$. Of course $H = \langle \sigma \rangle$ is a cyclic group of order p . As a K' -endomorphism of the K' -vector space L' , one gets that L' has an eigenspace decomposition $L' = L'_0 \oplus \dots \oplus L'_{p-1}$ with respect to σ , for which $L'_i = \{\ell' \in L' \mid \sigma(\ell') = \zeta^i \ell'\}$. Moreover, the contractions

Finally, to justify part (4) it suffices to argue that $\tau(s'_i) \in S'_j$ for some j , where $s'_i \in S'_i$. We mention that the initial eigenspace decomposition for σ is similarly an eigenspace decomposition for each σ^k . The equations $\tau\sigma\tau^{-1}(\tau s'_i) = \sigma^j(\tau s'_i)$ and $\tau\sigma\tau^{-1}(\tau s'_i) = \tau\sigma(s'_i) = \tau(\zeta^i s'_i) = \omega\tau(s'_i)$ where $\omega = \tau(\zeta^i) = \zeta^m$, for some m , show $\tau(s'_i) \in S'_k$ for some k , since we have argued $\tau(s'_i)$ is an eigenvector for σ^i . The observation here results from the fact $\langle\sigma\rangle$ is normal in G and the $\{\zeta^i\}_{i=0}^{p-1}$ are all conjugate under the action of G .

Theorem 2.3 (Notation as above). *We define S in A by $S = A \cap S'$. Then S is a free R -algebra in A such that $R \subseteq S$ and $pA \subseteq S$.*

Proof. Since S' is invariant as a set under the action of the entire Galois group G we see that $S' \cap A = S$ is invariant under the subgroup that corresponds to A . Therefore one sees that $t(S') = S$, where $t : A' \rightarrow A$ is the trace map. Since $[A' : A] = p - 1$ represents a unit in R we actually get that S is an S -direct summand of S' ; thus S is a free R -module. Finally, we observe that $pA \subseteq S$ since $pA' \subseteq S'$.

Theorem 2.4 (Koh's Theorem [22]). *Notation as above. $R \hookrightarrow A$ is necessarily R -split.*

Proof. Koh's result now follows from Proposition 2.1 where the R -free module F is taken to be the R -subalgebra S that is described in Theorem 2.3 and where the element “ x ” is taken to be $x = p$. We note that $R/pR \rightarrow A/pA$ is injective since R is integrally closed (see the proof of Theorem 2.1 (i)) and that $R/pR \rightarrow A/pA$ is R/pR -split since the equicharacteristic case of the direct summand conjecture is known to be true (see Hochster's article [15]).

Remark: When $\sqrt[p]{u}$ is replaced by $\sqrt[p^n]{u}$ one constructs, in the spirit above, a Kummer extension L'/K' where ζ in this case represents a primitive p^n th root of unity. Thus one obtains a free R' -subalgebra S' in the same way. However, technical problems arise when one contracts S' to A since $[A' : A] = [R' : R] = \varphi(p^n)$ is divisible by p for $n > 1$. In addition one merely gets that $p^n A' \subseteq S'$ and likewise $p^n A \subseteq S$ (one does not know $R/p^n R \rightarrow A/p^n A$ is split).

SECTION 3

As noted in the introduction, a cornerstone for constructing a proof of the syzygy theorem as given in [10], [12, pp. 58, 59]—or see [2, pp. 370, 371] for a more recent treatment, is the improved new intersection theorem (“INIT”). In fact there is an important sequence of implications that one can derive from Hochster [18] and Evans-Griffith [12, pp. 56–58].

Theorem on Canonical element $\xRightarrow{\text{Hochster}}$ INIT $\xRightarrow{\text{Evans-Griffith}}$ Order Ideal Theorem for Syzygies of Finite Projective Dimension $\xRightarrow{\text{Evans-Griffith}}$ Syzygy Theorem.

The various conjectures/theorems cited in the above sequence of implications are described in precise terms in Theorems 3.1 and 3.3. Heitmann [14] recently established the direct summand theorem for local rings of dimension (\leq) 3. At first glance one might guess that a low dimensional result of this type would have little impact on the syzygy conjecture. In actual fact the implication of Heitmann’s result with respect to INIT allows us to establish a less obvious case of the syzygy conjecture; see Corollary 3.5.

We remind the reader of a few basic definitions and facts that are taken for the most part from [12]. Let M be a finitely generated module over a local ring (R, m_R) . Suppose $pd_R M < \infty$, i.e., suppose M has finite projective dimension, and let $F_\bullet \rightarrow M$ represent a finite free resolution of M . Then $\text{rank } M = \sum_i (-1)^i \text{rank } F_i$. The i th kernel Z_i in F_\bullet is called the i th syzygy module for M ; the notation $\text{syz } Z \geq i$ means that Z is at least an i th syzygy for some R -module. For $e \in M$ one defines the order ideal “ $O_M(e)$ ” by

$$O_M(e) = \{f(e) \mid f \in \text{Hom}_R(M, R)\}.$$

One observes that $e \in O_M(e)M$ when M is a free R -module, since $O_M(e)$ is generated by the coordinate projections evaluated at e in this case.

The usual statement of the improved new intersection theorem goes as follows.

Theorem 3.1 (INIT; see [17] or [12, Theorem 1.13]). *Let F_\bullet be a finite free complex over the equicharacteristic local ring (R, m_R) such that*

- i) *length $H_i(F_\bullet) < \infty$ for $i > 0$, and*
- ii) *there is $e \in H_0(F_\bullet) - m_R H_0(F_\bullet)$ where $m_R^t e = 0$ for $t \gg 0$.*

Then $\text{length}(F_\bullet) \geq \dim R$.

Remark 3.2: Our application of INIT requires a slightly more special form. Namely, our complex will have the additional property that F_\bullet is locally trivial on $X_R = \text{Spec } R - m_R$, that is, $H_0(F_\bullet)$ is a locally free module on X_R . The fact that for validity of INIT it is enough to prove this special form was demonstrated in [3]. In this case one gets that $\text{length}(F_\bullet) \geq \dim R - 1$ even when R is not equicharacteristic, e.g., when R is of mixed characteristic p where $\dim R/pR < \dim R$. This observation follows from the fact F_\bullet/pF_\bullet satisfies the conditions of Theorem 3.1 over R/pR .

Theorem 3.1 and Remark 3.2 allow us to prove the order ideal theorem for syzygy modules of finite projective dimension as stated next (see a more general version [12, Theorem 3.14]).

Theorem 3.3. *Let (R, m_R) be a catenary local ring of dimension $n > 0$ and suppose that INIT holds for all homomorphic images of R having dimension not exceeding $\ell + 1$, where $\ell > 0$. Suppose E is a finitely generated nonfree R -module that is locally free on X_R . If $\text{pd}E < \infty$ and if $e \in E - m_RE$, then*

$$\text{codim } O_E(e) \geq n - \ell$$

when $\text{syz } E \geq n - \ell$.

Proof. The argument given here is much the same as the ones given in [10]. We suppose E satisfies the required hypothesis as stated above. The Auslander-Buchsbaum Theorem [1] provides the inequalities

$$\text{pd}E + \text{syz } E \leq \text{pd}E + \text{depth } E \leq \dim R = n.$$

Therefore, $\text{pd}E \leq n - (n - \ell) = \ell$. Let $F_\bullet \rightarrow E$ be a minimal R -free resolution of E . Since R is catenary one can show, for $I = O_E(e)$, that $\text{codim } I \geq n - \ell$ by establishing $\ell \geq \dim R/I$. Basechanging to the factor ring $R/I = \bar{R}$ gives a finite free \bar{R} -complex $\bar{F}_\bullet = F_\bullet/IF_\bullet$ that satisfies the hypothesis of INIT (3.1), since $\text{length}(\text{Tor}_i^R(R/I, E)) < \infty$, for $i > 0$, and since $\text{Supp}(R\bar{e}) = \{m_R\}$, where $\bar{e} = e + IE$. Remark 3.2 applies in this context; therefore

at worst one has $\ell \geq \text{length}(F_\bullet) \geq \dim \overline{R} - 1$ or what is the same: $\ell + 1 \geq \dim R/I$. Thus, by our assumption that INIT holds for homomorphic images of R with dimension $\leq \ell + 1$, we conclude that $\ell \geq \dim R/I$.

Corollary 3.4. *Let (R, m_R) be a local ring of dimension n and let ℓ be a positive integer such that all homomorphic images of S with dimension $\leq \ell + 1$ have INIT, where S is the completion of any R -algebra essentially of finite type. If E is a nonfree k th syzygy of finite projective dimension, where $k \geq n - \ell$, then $\text{rank } E \geq k$.*

Proof. Here our argument is rather similar to the one given in [11, pp. 7–10]. As before we may assume that E is locally free on X_R (via localization) and that R is complete. There is no harm in assuming $\text{syz } E = k = n - \ell$. By Theorem 3.3 we know $\text{codim } O_E(e) \geq n - \ell$. Therefore, if $n - \ell > \text{rank } E$, then we contradict the “Lemma” [11, p.7] that there is a minimal generator e in E (after possibly a finite residue field extension) with $\text{codim } O_E(e) \leq \text{rank } E$.

Although the conditions of Corollary 3.4 appear rather technical, the direct summand result of Heitmann [14] for local rings of dimension 3 together with Hochster’s result that the direct summand conjecture \Rightarrow INIT [17, Section 2] show that the conditions of Corollary 3.4 are valid for $\ell = 2$ (“every” local ring of dimension $\leq 3 = 2 + 1$ has the property INIT).

Corollary 3.5. *Let (R, m_R) be a local ring of dimension n . If E is a nonfree k th syzygy of finite projective dimension such that $k \geq n - 2$, then $\text{rank } E \geq k$.*

Proof. Apply Corollary 2.4 with $\ell = 2$.

Corollary 3.6. *The syzygy theorem holds for all regular local rings of dimension ≤ 5 .*

proof. The first serious case one must confront is that of $\text{syz } E \geq 2$ for which $\text{rank } E = 1$. However, such a module E is isomorphic to a reflexive ideal. Therefore $E \cong R$ since R is a UFD. The remaining case of consequence is that when $\text{syz } E \geq 3$ while $\text{rank } E = 2$. One observes this case is covered by Corollary 3.5. Any additional case would have $\text{pd } E \leq 1$, which has been known since the initial statement of the problem (see [11]).

As long as one restricts to modules of finite projective dimension, one can replace the regular ring R above by any integrally closed local domain (dimension $R \leq 5$) and 3.6 remains true.

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