A CONNECTION BETWEEN TWO SETS OF CONJECTURES

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In this article we are going to provide a brief survey of results on Serre's conjectures on intersection multiplicities, Hochster's canonical element conjecture and a connection between these two sets of conjectures as well. In particular we will formulate a new connection between canonical element conjecture and a special case of a question involving intersection multiplicity and χ_2-property of Serre. We will provide sketches of proofs of some of our statements in this short article. The details and related new results are going to be published in separate papers. Our main aim is to draw readers' attention a) to the current status of and recent work on these conjectures and b) to a relation between two apparently completely unrelated sets of problems.

Section 1

Serre's Conjecture on Intersection Multiplicity.

Let (R, m, K) be a regular local ring i.e. R is a regular local ring, m is its maximal ideal and $K = R/m$ is the residue field. Let M and N be two finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$ ("ℓ" stands for length). In [Se] Serre defined $\chi(M, N)$ as $\sum (-1)^j \ell(\mathrm{Tor}_j^R(M, N))$ and conjectured the following:

i) $\chi^R(M, N) \geq 0$, and

ii) $\chi^R(M, N) = 0$ if and only if $\dim M + \dim N < \dim R$.

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This is known as Serre's conjecture on intersection multiplicity. Note that this conjecture makes sense on any local ring provided one of the modules has finite projective dimension over the local ring. We drop "R" from the notation when there is no scope of ambiguity. Serre proved the conjecture for equicharacteristic and unramified regular local rings (theorem 1, theorem 2, ch. V, [Se]). He also proved the following results in the same chapter: Let R, M, N be as above and let $\ell(M \otimes_R N) < \infty$. Then

a) $\dim M + \dim N \leq \dim R$ (theorem 3, ch. V, [Se]),

b) if $i = \dim R - \operatorname{depth} M - \operatorname{depth} N$, then $\operatorname{Tor}_j(M, N) = 0$ for $j > i$ (theorem 4, ch. V, [Se]), and

c) if $\dim M + \dim N = \dim R$, and R is equicharacteristic, then $\chi(M, N) \geq e_m(M)e_m(N)$ where $e_m(T)$ denotes the Hilbert-Samuel multiplicity of a finitely generated R-module T (complement, theorem 1, ch. V, [Se]).

This conjecture can be divided into three parts:

(i) Vanishing: $\chi(M, N) = 0$ when $\dim M + \dim N < \dim R$,

(ii) Non-negativity: $\chi(M, N) \geq 0$, and

(iii) Positivity: $\chi(M, N) > 0$ when $\dim M + \dim N = \dim R$.

P. Roberts [R1], H. Gillet and C. Soulé [G-S] proved the vanishing part independently (in mid-eighties). Their proofs extend to the local complete intersections when both M and N have finite projective dimension. The hope of generalizing the validity of this conjecture to non-regular rings when only one of the modules has finite projective dimension was dashed by a counterexample due to Hochster, McLaughlin and this author [D-H-M] in the early eighties. This example also led to counter-examples to several other multiplicity related conjectures. In the mid-nineties Gabber ([B], [G]) proved the non-negativity part of the conjecture. The positivity part of Serre's conjecture has been open for almost fifty years. The fact that positivity or non-negativity implies vanishing was proved in [D1] in the early eighties.

In the mid-nineties Gabber ([G]) came up with a brilliant idea to prove part i) of this conjecture. The key steps in his proofs are the following: (a) use de Jong's theorem on regular alteration to reduce $\chi(R/P, R/q)$ to $\chi^{\mathcal{O}_{X'}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'})$, where $X' = \mathbb{P}^N_R$, Y', Z'

are closed subvarieties of X' such that Z' is regular and support of $\mathrm{Tor}_i^{\mathcal{O}_{X'}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'}) \subset \mathbb{P}_K^N$ for $i \leq 0$, (b) extend a spectral sequence argument of Serre to reduce the problem to χ of intersection of a closed subscheme of the normal bundle corresponding to the regular imbedding $Z' \hookrightarrow X'$ with the 0-section of the bundle, (c) use ramification of R and module of the differentials to show that the closed fiber E over $s = [m]$ of this bundle is generated by global sections (highly ingenious technique!) and (d) establish non-negativity of intersection multiplicity on vector bundles over projective schemes generated by global sections.

This work has energized the study of $\chi^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, on a regular scheme X essentially of finite type over a field on an excellent discrete valuation ring, where $\mathcal{F}, \mathcal{G}$ are coherent $\mathcal{O}_X$-modules such that $\ell(H^i(X, \mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))) < \infty$. Inspired by Gabber's work we started studying intersection multiplicity for both local and global set-up. First we observed the following result.

Result 1 (Theorem 2, D9) R as above and, let P, q be two prime ideals of R such that $\ell(R/(P+q)) < \infty$. Write $X = \operatorname{Spec} R$, $Y = \operatorname{Spec}(R/P)$, $Z = \operatorname{Spec}(R/q)$. Let $\pi : \widetilde{X} \to X$ be the blow-up of X at $\{m\}$, $\mathcal{E}$ the exceptional divisor and $\eta : \mathcal{E} \to \{m\}$ the induced map. Since R is regular local of dimension n, $\mathcal{E} = \mathbb{P}_K^{n-1}$. Let $\widetilde{Y}$, $\widetilde{Z}$ denote the blow-ups of Y and Z respectively at $\{m\}$. Since $\ell(R/(P+q)) < \infty$, $\widetilde{Y} \cap \widetilde{Z} \subset \mathcal{E}$. Hence $\ell(H^i(\widetilde{X}, \mathrm{Tor}_j^{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Z}})))$ is finite for i, $j \geq 0$. For any finitely generated R-module M, $e_m(M)$ denotes the Hilbert-Samuel multiplicity of M and for any pair M, N of such modules, $P_{\chi(G(M),G(N))}$ denotes the alternating sum of Hilbert polynomials $P_{\mathrm{Tor}_i(G(M),G(N))}(t)$, for $t \gg 0$, on the graded ring $G(R)$. Now we state our theorem.

Theorem. *With the above set-up we have the following:*

(i) *if* $\dim R/P + \dim R/q = \dim R$, *then*

$$\chi(R/P, R/q) = \chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) + P_{\chi(G(R/P),G(R/q))}$$

and

(ii) *if* $\dim R/P + \dim R/q < \dim R$, *then*

$$\chi(R/P, R/q) = \chi^{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Z}}).$$

As a Corollary, first we deduce that in case (i), $\chi(R/P, R/q) = e_m(R/P)e_m(R/q) + \chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}})$ and then we deduce, via Serre's Theorem (result (c)), that when R is equicharacteristic, $\chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) \geq 0$. This Corollary has been mentioned in Fulton's book (Example 20.4.3, [Fu]). Here, our theorem connects local and global χ directly via the tangent cone and the blow-up $\tilde{X}$ of the local ring without using any heavy machinery and it formulates what happens in the vanishing part which is completely new.

Recall that in order to prove non-negativity Gabber required the vector bundle E over a projective scheme to be generated by global sections (step (d)). In our next result we describe what happens over any vector bundle and deduce that vanishing holds even when the bundle is not generated by global sections.

Result 2 (Proposition 2.2 [D9]). *Let W be a projective scheme over a field and $\mathcal{L}$ be a locally free $\mathcal{O}_W$-module of rank d. Let $E = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_W}(\mathcal{L}))$ and let V be a closed subscheme E. Let $\beta : W \to E$ denote the $\mathcal{O}$-section; we identify W with $\beta(W)$ when there is no scope of ambiguity. We have the following:*

i) $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = \int_{\overline{V}} c_d(\xi_{\overline{V}}) td(\xi_{\overline{V}})^{-1}\tau(\overline{V})$, where $\xi_{\overline{V}}$ is the restriction of the universal quotient bundle ξ on $\mathbb{P}(E \bigoplus 1)$ to the projective closure $\overline{V}$ of V;

ii) if $\dim V < d$, then $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = 0$;

iii) if $V \cap W = \phi$, then $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = 0$;

and

iv) if $\dim V = d$, then $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_W) = \int_{\overline{V}} c_d(\xi_{\overline{V}})[\overline{V}] = \int \beta^*([V])$.

Part iv) of the above proposition along with Theorem 12.1(a) in [Fu] provide a new proof of Gabber's proposition on non-negativity over vector bundles generated by global sections.

Result 3. We use the above proposition and Gabber's technique to prove our next theorem. Part a) of this theorem can also be deduced by using intersection cycles, diagonalization and Riemann-Roch Theorem ([Fu-M]).

Theorem (Theorem 3, [D9]). *Let X be a smooth projective variety over a perfect field k. Let Y and Z be two closed subvarieties of X such that $\dim Y + \dim Z \leq \dim X$. We have* *the following:*

a) If $\dim Y + \dim Z < \dim X$*, then* $\chi^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$.

b) if T_X *is generated by global sections and* $\dim Y + \dim Z = \dim X$*, then* $\chi^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \geq 0$*. Here* T_X *is the tangent bundle on* X*.*

In our next result both vanishing and non-negativity are established on $\mathbb{P}^n_R$, R as above.

Result 4 (Theorem 4 [D9])**.** *Let* Y *and* Z *be two closed subschemes of* $\mathbb{P} = \mathbb{P}^d_R$ *such that* $Y \cap Z \subset \mathbb{P}^d_s = \mathbb{P}^d_K$*. Then*

(i) if $\dim Y + \dim Z < \dim \mathbb{P}$*,* $\chi^{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$

and (ii) if $\dim Y + \dim Z = \dim \mathbb{P}$*,* $\chi^{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_Y, \mathcal{O}_Z) \geq 0$*.*

The final result of this section is concerned with intersection multiplicity over $\tilde{X}$, the blow-up of $X = \mathrm{Spec}(R)$, R regular local essentially of finite type over a field or an excellent discrete valuation ring. Let $\tilde{Y}, \tilde{Z}$ be two closed subvarieties of $\tilde{X}$ such that $\tilde{Y} \cap \tilde{Z} \subset \mathbb{P}^{n-1}_K$ (= the fiber over $[m]$). Result 2 plays an important role in the proof of part i) of this result.

Result 5 (parts i) and ii) of Theorem 5, [D9]). Let $\tilde{X}, \tilde{Y}, \tilde{Z}$ be as above. Suppose that $\dim \tilde{Y} + \dim \tilde{Z} \leq \dim \tilde{X}$. We have the following:

i) if $\dim \tilde{X} + \dim \tilde{Z} < \dim \tilde{X}$ or $\tilde{Y} \cap \tilde{Z} = \phi$, then $\chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) = 0$, and

ii) Let $\tilde{Y}, \tilde{Z}$ be blow-ups of Y and Z (notations in Result 2). If $\dim(G(R/P) \otimes G(R/q)) \leq 1$, then $\chi^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\tilde{Z}}) \geq 0$.

As a corollary (Result 1 and Result 5) we derive a new proof of Serre-vanishing and a proof of a special case of positivity i.e., $\chi(M,N) \geq e_m(M)e_m(N)$, if $\dim(G(M) \otimes G(N)) \leq 1$. In the late seventies Tennison [T] proved that $\chi(M,N) = e_m(M)e_m(N)$ if $\dim(G(M) \otimes G(N)) = 0$.

χ_i-Conjecture of Serre.

Let R be a regular local ring and let M and N be two finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$. Serre defined $\chi_i^R(M,N)$ for $i > 0$, as $\sum (-1)^j \ell(\mathrm{Tor}^R_{i+j}(M,N))$,

$j \geq 0$ and conjectured the following: i) $\chi_i^R(M,N) \geq 0$ and ii) $\chi_i^R(M,N) = 0$ if and only if $\mathrm{Tor}_j^R(M,N) = 0$ for $j \geq i$.

This conjecture too makes sense over any local ring provided one of the modules has finite projective dimension over the local ring. We drop "R" from the notation when there is no scope of ambiguity.

Serre [Se] proved the conjecture in the equicharacteristic case. Lichtenbaum [L] proved it over unramified regular local rings for $i \geq 2$; he also showed that when both M and N are torsion free the conjecture is valid for $i = 1$. Later Hochster [Ho3] gave a complete proof for this case. The conjecture is very much open over ramified regular local rings.

We have been studying this conjecture and its relation with intersection multiplicity for quite some time. The following result was proved in [D4]:

Theorem (Theorem 3.2, [D4])**.** *Let R be a regular local ring of dimension n and let M be a Cohen-Macaulay module over R. Then for any module N with $\ell(M \otimes N) < \infty$ and $\dim M + \dim N = \dim R$, $\chi(M,N)$ is positive if for every pair of modules (M', N') such that $\dim M' + \dim N' < \dim R$ and $\ell(M' \otimes N') < \infty$, $\chi_2(M', N') \geq 0$.*

This theorem assures positivity when one of the modules is Cohen-Macaulay, provided χ_2 is non-negative in the vanishing set-up. The usual expectation was that whenever vanishing holds, positivity and χ_i-conjectures should also be valid. Recall that in [D-H-M] the counter-example to vanishing over a hypersurface, when only one of the modules had finite projective dimension, led to counter-examples for both positivity and χ_i-conjecture in the above set-up.

In [D-H-M] Hochster, McLaughlin and this author constructed modules M of finite length and finite projective dimension over a 3-dimensional local hypersurface $R(= k[x,y,u,v]_m/(xy - uv)$, m being the maximal ideal generated by x, y, u and v) with negative intersection multiplicity. In this case $\chi(M, R/P) = -1$ where $P = (x,u)$ and thus vanishing fails. Afterwards Levine [L] extended it to higher dimensions (e.g., $R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]_m/\Sigma x_i y_i$) by using K-theoretic techniques. Recently Roberts and Srinivas [R-S] proved the existence of many such examples of hypersurfaces (smooth cubic-surfaces in $\mathbb{P}^3$) and Gorenstein rings (coordinate rings of $\mathbb{P}^n \times \mathbb{P}^n$) by using local Chern characters and the localization exact sequence from K-theory of Thomason and Trobaugh. The following results were proved in [D8].

Result 1. *All the hypersurfaces, mentioned above, possess pairs of modules M and N, both having finite projective dimension, such that $\chi(M,N)=0$ and $\chi_2(M,N)<0$.*

Result 2. *The conclusion of Result 1 is valid on any 3-dimensional Gorenstein ring where generalized vanishing (i.e. when only one module has finite projective dimension) fails.*

To be honest, it was rather shocking to us as we, like many other researchers, expected χ_i-conjecture to be valid in the above situation since vanishing part of intersection multiplicity is valid in such set-up. Due to the theorem stated earlier, on relation between positivity and χ_2, *We now suspect that positivity may not be valid over complete intersections even when both modules have finite projective dimension.*

Section 2

Canonical Element Conjecture of M. Hochster (henceforth CEC).

Let (A,m,K) be a noetherian local ring (henceforth "local" will mean "noetherian local") of dimension n and let S_i denote the i^{th} syzygy of the residue field K. Let $\theta_n : \operatorname{Ext}^n_A(K,S_n) \to H^n_m(S_n)$ denote the direct limit map (Recall that for any module M, $H^i_m(M) = \varinjlim_{t} \operatorname{Ext}^i_A(A/m^t, M))$ and let $\eta_A = \theta_n$ (class of identity map on S_n). Hochster called η_A the "canonical element" of the local ring A and conjectured that $\eta_A \neq 0$ for every such A [Ho2].

In elementary terms this conjecture asserts the following:

For every free resolution $F_\bullet : \cdots \to A^{s_i} \to A^{s_{i-1}} \to \cdots \to A^{s_0} \to K \to 0$ of K and for every system of parameters $x_1,\ldots,x_n$ of A, if $\phi_\bullet$ is any map from the Koszul complex $K_\bullet(\underline{x};A)$ (for the above system of parameters) to $F_\bullet$, lifting the natural surjection $A/(x_1,\ldots,x_n)$ to K, then $\phi_n : K_n(\underline{x};A) \to A^{s_n}$ is non-zero.

In [Ho2] Hochster introduced several equivalent forms of this conjecture and proved it for equicharacteristic local rings. One of the earliest forms, the direct summand conjecture was proved a decade earlier for the same class of rings. The conjecture states the following:

Direct Summand Conjecture (henceforth DSC) [Ho2]**.**

Let R be a regular local ring and let $i : R \hookrightarrow A$ be a module-finite extension of R (i.e. A is an R-algebra such that A is also a finitely generated R-module). Then i splits as an R-module map.

The geometers were aware of the validity of this conjecture in characteristic 0 for module finite extensions over normal domains.

An analysis of this conjecture leads to the following:

Monomial Conjecture (henceforth MC) [Ho1]**.**

Let A be a local ring of dimension n and let $x_1, \ldots x_n$ be a system of parameters of A. Then, for every integer $t > 0$,

$$(x_1 \ldots x_n)^t \notin (x_1^{t+1}, \ldots, x_n^{t+1}).$$

In 1980 Evans and Griffith [E-G], in their proof of the syzygy problem, made a crucial observation in the equicharacteristic case which can be viewed as an extension of the intersection theorem. Hochster introduced this observation as the following conjecture in [H2]:

Improved New Intersection Conjecture ([E-G]**,** [Ho2]**) (henceforth INIC).**

Let A be a local ring. Let $F_\bullet$ be a complex of finitely generated A-modules

$$F_\bullet : 0 \to F_s \to \cdots \to F_1 \to F_0 \to 0$$

such that $\ell(H_i(F_\bullet)) < \infty$ for $i > 0$ and $H_0(F_\bullet)$ has a minimal generator annihilated by a power of m. Then $\dim A \leq s$.

In [Ho2] Hochster pointed out that CEC implies INIC. In [D2] we proved the reverse implication and thus CEC is equivalent to INIC. Over the years several special cases of the canonical element conjecture has been proved and new equivalent forms have been introduced ([D2], [D5], [D6], [D7], [Go], [Hei], [K]). Heitmann's [Hei] proof of DSC for dimension 3 is the most notable among these results. Goto [Go] proved MC for Buchsbaum rings and Koh [K] proved DSC for degree p extensions. The following results play an important role in the study of CEC from the perspective presented in this article.

a) Let S_i denote the i^{th} syzygy of $K(= A/m)$ in a minimal free resolution of $A : \dim A = n$. Let η_i = image of (class of identity map on S_i) under the direct limit map $\theta_i : \operatorname{Ext}^i_A(K, S_i) \to H^i_m(S_i)$. Then $\eta_i \neq 0$ for $0 \leq i \leq n-1$ ([D7]).

Recall that CEC demands θ_n to be non-zero.

b) Let A be a complete local normal domain of dimension n. Write $A = S/P$ where S is a complete intersection such that $\dim S = \dim A$. Let $\Omega = \operatorname{Hom}_S(A, S)$—the canonical module for A. Then S/Ω satisfies CEC ([D7]).

c) (Strooker and Stückrad) Notations as in b). MC is valid over A if and only if for every system of parameters $x_1, \ldots x_n$ of S, $\Omega \not\subset (x_1, \ldots x_n)$.

d) MC is valid for all local rings if and only if MC is valid for all local almost complete intersection rings. [D6]

e) Let A be a local almost complete intersection ring and let $x_1, \ldots x_n$ be a system of parameters of A. Then $x_1, \ldots x_n$ satisfies MC if and only if $\ell(H_1(\underline{x}; A)) < \ell(A/\underline{x})$; here $\underline{x}$ denotes the ideal generated by $x_1, \ldots, x_n$ ([D6]) as well.

Section 3

Connection with Intersection Multiplicity and χ_i-Property.

Given a local ring A and a pair of finitely generated modules M and N such that $\ell(M \otimes_A N) < \infty$, we ask the following question (Q): is $\ell(M \otimes_A N) > \ell(\operatorname{Tor}_1(M, N))$? it is clear that (Q) has obvious negative answers, e.g., when $M = N = K$—the residue field of A or when $M = K$, $N = m$, the maximal ideal of A. This follows from the fact that $\operatorname{Tor}^A_i(K, K) \geq \binom{s}{i}$ where s = minimal number of generators of m ([Se]). To investigate the question in proper perspective we assume that A is a regular local ring. In this case (Q) boils down to the following: is $\chi(M, N) > \chi_2(M, N)$? Now the whole machinery of intersection multiplicity and χ_i-property mentioned in section 1 come into play. We have the following preposition:

3.1 Proposition. *Let R be an equicharacteristic or unramified regular local ring and let M, N be two finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$. We have the* *following:*

i) if $\dim M + \dim N < \dim R$ *and* $\operatorname{depth} M + \operatorname{depth} N = \dim R - 1$*, then* $\ell(M \otimes_R N) = \ell(\operatorname{Tor}_1^R(M, N))$.

ii) if $\dim M + \dim N < \dim R$ *and* $\operatorname{depth} M + \operatorname{depth} N < \dim R - 1$*, then* $\ell(M \otimes_R N) < \ell(\operatorname{Tor}_1^R(M, N))$*;*

iii) if $\dim M + \dim N = \dim R$ *and* $\operatorname{depth} M + \operatorname{depth} N = \dim R - 1$*, then* $\ell(M \otimes_R N) > \ell(\operatorname{Tor}_1^R(M, N))$.

Proof. By definition we have $\ell(M \otimes N) - \ell(\operatorname{Tor}_1(M, N) = \chi(M, N) - \chi_2(M, N)$.

For i), we have $\chi(M, N) = 0$ and $\operatorname{Tor}_i(M, N) = 0$ for $i \geq 2$ (vanishing + result b) in section 1);

for ii), $\chi(M, N) = 0$ and $\chi_2(M, N) > 0$ (vanishing + Serre/Lichtenbaum's result on χ_i);

and for iii), $\chi(M, N) > 0$ and $\operatorname{Tor}_i(M, N) = 0$ for $i \geq 2$ (positivity + result b) in section 1).

The above proposition shows that (Q) has a definite answer in the vanishing set-up i.e. when $\dim M + \dim N < \dim R$. The problem begins when we consider the positivity set-up i.e. when $\dim M + \dim N = \dim R$. Even when $N = R/I$ and I is generated by an R-sequence, i.e. I is a complete intersection ideal, any definite answer to (Q) is extremely difficult to guess.

3.2 The theorem in this section demonstrates why a definite answer to (Q), even when one of the modules is of the form R/I where I is generated by an R-sequence, is so difficult to comprehend. This is accomplished by connecting a special case of (Q) to the monomial conjecture. Recall that an ideal J of R is called an almost complete intersection ideal if J is minimally generated by $(htJ + 1)$ elements and in such a situation we call R/J an almost complete intersection ring.

Theorem. *MC is valid for all local rings A if and only if for any regular local ring R and for every pair of ideals* I, J *of R such that i) I is a complete intersection ideal (i.e. generated by an R-sequence), ii) J is an almost complete intersection ideal, iii)* $htI + htJ = \dim R$

and iv) $I + J$ is m-primary, $\ell(R/(I+J)) > \ell(\mathrm{Tor}_1(R/I, R/J))$ i.e. (Q) has a positive answer for the pair $(R/I, R/J)$.

Proof. By result d) in section 2, for validity of MC we can assume that local ring $A = S/\lambda S$, where λ is a zero-divisor in a complete intersection ring S; $S = R/(y_1, \ldots, y_r)$ where R is a regular local ring and $\{y_1, \ldots, y_r\}$ is an R-sequence. Consider a system of parameters $x_1, \ldots, x_n$ of A. Then, by result e) in section 2, MC is valid over A if and only if $\ell(H_1(\underline{x}; A)) > \ell(A/\underline{x})$, $\underline{x}$ denoting the ideal generated by $x_1, \ldots, x_n$.

We can lift $x_1, \ldots, x_n$ to a system of parameters $x'_i, \ldots, x'_n$ of S and next to a part of a system of parameters $x''_i, \ldots, x''_n$ of R. Hence $\{x''_1, \ldots, x''_n\}$ form an R-sequence. Write $I = (x''_1, \ldots, x''_n)$ and $J = (y_1, \ldots, y_r, \lambda)$. Then $\mathrm{Tor}_i^R(R/I, R/J) = \mathrm{Tor}_i^S(S/\underline{x}', S/\lambda S) = H_1(\underline{x}; A)$ for $i \geq 0$. Hence MC is valid over A if and only if $\ell(R/(I+J)) > \ell(\mathrm{Tor}_i^R(R/I, R/J))$.

Conversely, given I, J as in the statement of the theorem we can retrace the above steps to prove our assertion.

Remarks.

1. For the validity of MC we can assume R is unramified/equicharacteristic.
2. R, I, J etc. as in the statement of the above theorem; then $\ell(R/(I+J)) \geq \ell(\mathrm{Tor}_1^R(R/I, R/J))$.

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