EXERCISES 3.1

GOAL *Use the concepts of the image and the kernel of a linear transformation (or a matrix). Express the image and the kernel of any matrix as the span of some vectors. Use kernel and image to determine whether a matrix is invertible.*

For each matrix A in Exercises 1 through 13, find vectors that span the kernel of A. Use paper and pencil.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

9. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$

13. $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

For each matrix A in Exercises 14 through 16, find vectors that span the image of A. Give as few vectors as possible. Use paper and pencil.

14. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

16. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

For each matrix A in Exercises 17 through 22, describe the image of the transformation $T(\vec{x}) = A\vec{x}$ geometrically (as a line, plane, etc. in $\mathbb{R}^2$ or $\mathbb{R}^3$).

17. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

21. $A = \begin{bmatrix} 4 & 7 & 3 \\ 1 & 9 & 2 \\ 5 & 6 & 8 \end{bmatrix}$

22. $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix}$

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y = x/3$ in $\mathbb{R}^2$

24. Orthogonal projection onto the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$

25. Rotation through an angle of $\pi/4$ in the counterclockwise direction (in $\mathbb{R}^2$)

26. What is the image of a function f from $\mathbb{R}$ to $\mathbb{R}$ given by

$$f(t) = t^3 + at^2 + bt + c,$$

where a, b, c are arbitrary scalars?

27. Give an example of a noninvertible function f from $\mathbb{R}$ to $\mathbb{R}$ with $\text{im}(f) = \mathbb{R}$.

28. Give an example of a parametrization of the ellipse

$$x^2 + \frac{y^2}{4} = 1$$

in $\mathbb{R}^2$. (See Example 3.)

29. Give an example of a function whose image is the unit sphere

$$x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^3$.

30. Give an example of a matrix A such that $\text{im}(A)$ is spanned by the vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

31. Give an example of a matrix A such that $\text{im}(A)$ is the plane with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

32. Give an example of a linear transformation whose image is the line spanned by

$$\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

in $\mathbb{R}^3$.

33. Give an example of a linear transformation whose kernel is the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

34. Give an example of a linear transformation whose kernel is the line spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^3$.

35. Consider a nonzero vector $\vec{v}$ in $\mathbb{R}^3$. Arguing geometrically, describe the image and the kernel of the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$ given by

$$T(\vec{x}) = \vec{v} \cdot \vec{x}.$$

36. Consider a nonzero vector $\vec{v}$ in $\mathbb{R}^3$. Arguing geometrically, describe the image and the kernel of the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$T(\vec{x}) = \vec{v} \times \vec{x}.$$

See Definition A.9 in the Appendix.

37. For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

describe the images and kernels of the matrices A, A^2, and A^3 geometrically.

38. Consider a square matrix A.
   a. What is the relationship between $\ker(A)$ and $\ker(A^2)$? Are they necessarily equal? Is one of them necessarily contained in the other? More generally, what can you say about $\ker(A)$, $\ker(A^2)$, $\ker(A^3)$, $\ker(A^4)$, ...?
   b. What can you say about $\operatorname{im}(A)$, $\operatorname{im}(A^2)$, $\operatorname{im}(A^3)$, ...?

   (*Hint*: Exercise 37 is helpful.)

39. Consider an $n \times p$ matrix A and a $p \times m$ matrix B.
   a. What is the relationship between $\ker(AB)$ and $\ker(B)$? Are they always equal? Is one of them always contained in the other?
   b. What is the relationship between $\operatorname{im}(A)$ and $\operatorname{im}(AB)$?

40. Consider an $n \times p$ matrix A and a $p \times m$ matrix B. If $\ker(A) = \operatorname{im}(B)$, what can you say about the product AB?

41. Consider the matrix $A = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$.
   a. Describe $\ker(A)$ and $\operatorname{im}(A)$ geometrically.
   b. Find A^2. If $\vec{v}$ is in the image of A, what can you say about $A\vec{v}$?
   c. Describe the transformation $T(\vec{x}) = A\vec{x}$ geometrically.

42. Express the image of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 2 \\ 1 & 4 & 7 & 0 \end{bmatrix}$$

as the kernel of a matrix B. *Hint*: The image of A consists of all vectors $\vec{y}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{y}$ is consistent. Write this system more explicitly:

$$\left|\begin{array}{l} x_1 + x_2 + x_3 + 6x_4 = y_1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = y_2 \\ x_1 + 3x_2 + 5x_3 + 2x_4 = y_3 \\ x_1 + 4x_2 + 7x_3 = y_4 \end{array}\right|.$$

Now, reduce rows:

$$\left|\begin{array}{l} x_1 - x_3 + 8x_4 = 4y_3 - 3y_4 \\ x_2 + 2x_3 - 2x_4 = -y_3 + y_4 \\ 0 = y_1 - 3y_3 + 2y_4 \\ 0 = y_2 - 2y_3 + y_4 \end{array}\right|.$$

For which vectors $\vec{y}$ is this system consistent? The answer allows you to express $\operatorname{im}(A)$ as the kernel of a 2×4 matrix B.

43. Using your work in Exercise 42 as a guide, explain how you can write the image of any matrix A as the kernel of some matrix B.

44. Consider a matrix A, and let $B = \operatorname{rref}(A)$.
   a. Is $\ker(A)$ necessarily equal to $\ker(B)$? Explain.
   b. Is $\operatorname{im}(A)$ necessarily equal to $\operatorname{im}(B)$? Explain.

45. Consider an $n \times m$ matrix A with $\operatorname{rank}(A) = r < m$. Explain how you can write $\ker(A)$ as the span of $m - r$ vectors.

46. Consider a 3×4 matrix A in reduced row-echelon form. What can you say about the image of A? Describe all cases in terms of $\operatorname{rank}(A)$, and draw a sketch for each.

47. Let T be the projection along a line L_1 onto a line L_2. (See Exercise 2.2.33.) Describe the image and the kernel of T geometrically.

48. Consider a 2×2 matrix A with $A^2 = A$.
   a. If $\vec{w}$ is in the image of A, what is the relationship between $\vec{w}$ and $A\vec{w}$?
   b. What can you say about A if $\text{rank}(A) = 2$? What if $\text{rank}(A) = 0$?
   c. If $\text{rank}(A) = 1$, show that the linear transformation $T(\vec{x}) = A\vec{x}$ is the projection onto $\text{im}(A)$ along $\ker(A)$. (See Exercise 2.2.33.)

49. Verify that the kernel of a linear transformation is closed under addition and scalar multiplication. (See Theorem 3.1.6.)

50. Consider a square matrix A with $\ker(A^2) = \ker(A^3)$. Is $\ker(A^3) = \ker(A^4)$? Justify your answer.

51. Consider an $n \times p$ matrix A and a $p \times m$ matrix B such that $\ker(A) = \{\vec{0}\}$ and $\ker(B) = \{\vec{0}\}$. Find $\ker(AB)$.

52. Consider a $p \times m$ matrix A and a $q \times m$ matrix B, and form the block matrix

$$C = \begin{bmatrix} A \\ B \end{bmatrix}.$$

What is the relationship between $\ker(A)$, $\ker(B)$, and $\ker(C)$?

53. In Exercises 53 and 54, we will work with the binary digits (or bits) 0 and 1, instead of the real numbers $\mathbb{R}$. Addition and multiplication in this system are defined as usual, except for the rule $1 + 1 = 0$. We denote this number system with $\mathbb{F}_2$, or simply $\mathbb{F}$. The set of all vectors with n components in $\mathbb{F}$ is denoted by $\mathbb{F}^n$; note that $\mathbb{F}^n$ consists of 2^n vectors. (Why?) In information technology, a vector in $\mathbb{F}^8$ is called a *byte*. (A byte is a string of 8 binary digits.)

The basic ideas of linear algebra introduced so far (for the real numbers) apply to $\mathbb{F}$ without modifications.

A *Hamming matrix* with n rows is a matrix that contains all nonzero vectors in $\mathbb{F}^n$ as its columns (in any order). Note that there are $2^n - 1$ columns. Here is an example:

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{matrix} 3 \text{ rows} \\ 2^3 - 1 = 7 \text{ columns}. \end{matrix}$$

   a. Express the kernel of H as the span of four vectors in $\mathbb{F}^7$ of the form

$$\vec{v}_1 = \begin{bmatrix} * \\ * \\ * \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} * \\ * \\ * \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

   b. Form the 7×4 matrix

$$M = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

   Explain why $\text{im}(M) = \ker(H)$. If $\vec{x}$ is an arbitrary vector in $\mathbb{F}^4$, what is $H(M\vec{x})$?

54. (See Exercise 53 for some background.) When information is transmitted, there may be some errors in the communication. We present a method of adding extra information to messages so that most errors that occur during transmission can be detected and corrected. Such methods are referred to as *error-correcting codes*. (Compare these with codes whose purpose is to conceal information.) The pictures of man's first landing on the moon (in 1969) were televised just as they had been received and were not very clear, since they contained many errors induced during transmission. On later missions, much clearer error-corrected pictures were obtained.

In computers, information is stored and processed in the form of strings of binary digits, 0 and 1. This stream of binary digits is often broken up into "blocks" of eight binary digits (bytes). For the sake of simplicity, we will work with blocks of only four binary digits (i.e., with vectors in $\mathbb{F}^4$), for example,

$$\cdots \mid 1\ 0\ 1\ 1 \mid 1\ 0\ 0\ 1 \mid 1\ 0\ 1\ 0 \mid 1\ 0\ 1\ 1 \mid 1\ 0\ 0\ 0 \mid \cdots.$$

Suppose these vectors in $\mathbb{F}^4$ have to be transmitted from one computer to another, say, from a satellite to ground control in Kourou, French Guiana (the station of the European Space Agency). A vector $\vec{u}$ in $\mathbb{F}^4$ is first transformed into a vector $\vec{v} = M\vec{u}$ in $\mathbb{F}^7$, where M is the matrix you found in Exercise 53. The last four entries of $\vec{v}$ are just the entries of $\vec{u}$; the first three entries of $\vec{v}$ are added to detect errors. The vector $\vec{v}$ is now transmitted to Kourou. We assume that at most one error will occur during transmission; that is, the vector $\vec{w}$ received in Kourou will be either $\vec{v}$ (if no error has occurred) or $\vec{w} = \vec{v} + \vec{e}_i$ (if there is an error in the ith component of the vector).

   a. Let H be the Hamming matrix introduced in Exercise 53. How can the computer in Kourou use $H\vec{w}$ to determine whether there was an error in the transmission? If there was no error, what is $H\vec{w}$? If there was an error, how can the computer determine in which component the error was made?

Proof In Example 9 we have shown only one part of Theorem 3.2.10; we still need to verify that the uniqueness of the representation $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ (for every $\vec{v}$ in V) implies that $\vec{v}_1, \ldots, \vec{v}_m$ is a basis of V. Clearly, the vectors $\vec{v}_1, \ldots, \vec{v}_m$ span V, since every $\vec{v}$ in V can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$.

To show the linear independence of vectors $\vec{v}_1, \ldots, \vec{v}_m$, consider a relation $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$. This relation is a representation of the zero vector as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$. But this representation is unique, with $c_1 = \cdots = c_m = 0$, so that $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$ must be the trivial relation. We have shown that vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent. ■

Consider the plane $V = \text{image}(A) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ introduced in Example 4. (Take another look at Figure 4.)

We can write

$$\begin{aligned}\vec{v}_4 &= 1\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3 + 0\vec{v}_4 \\ &= 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 1\vec{v}_4,\end{aligned}$$

illustrating the fact that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ do not form a basis of V. However, every vector $\vec{v}$ in V can be expressed *uniquely* as a linear combination of $\vec{v}_1$ and $\vec{v}_3$ alone, meaning that the vectors $\vec{v}_1, \vec{v}_3$ do form a basis of V.

EXERCISES 3.2

GOAL ***Check whether or not a subset of $\mathbb{R}^n$ is a subspace. Apply the concept of linear independence (in terms of Definition 3.2.3, Theorem 3.2.7, and Theorem 3.2.8). Apply the concept of a basis, both in terms of Definition 3.2.3 and in terms of Theorem 3.2.10.***

Which of the sets W in Exercises 1 through 3 are subspaces of $\mathbb{R}^3$?

1. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$

2. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \le y \le z \right\}$

3. $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

4. Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$. Is $\text{span}(\vec{v}_1, \ldots, \vec{v}_m)$ necessarily a subspace of $\mathbb{R}^n$? Justify your answer.

5. Give a geometrical description of all subspaces of $\mathbb{R}^3$. Justify your answer.

6. Consider two subspaces V and W of $\mathbb{R}^n$.

a. Is the intersection $V \cap W$ necessarily a subspace of $\mathbb{R}^n$?

b. Is the union $V \cup W$ necessarily a subspace of $\mathbb{R}^n$?

7. Consider a nonempty subset W of $\mathbb{R}^n$ that is closed under addition and under scalar multiplication. Is W necessarily a subspace of $\mathbb{R}^n$? Explain.

8. Find a nontrivial relation among the following vectors:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

9. Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$, with $\vec{v}_m = \vec{0}$. Are these vectors linearly independent?

In Exercises 10 through 20, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

10. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

11. $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

12. $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

13. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

15. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

17. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

18. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$

19. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$

20. $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}4\\5\\0\end{bmatrix}, \begin{bmatrix}6\\7\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

In Exercises 21 through 26, find a redundant column vector of the given matrix A, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of A. (This procedure is illustrated in Example 8.)

21. $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ **22.** $\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}$. **23.** $\begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}$

24. $\begin{bmatrix}1 & 3 & 6\\1 & 2 & 5\\1 & 1 & 4\end{bmatrix}$ **25.** $\begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\1 & 0 & 1\end{bmatrix}$

26. $\begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 3 & 0\\0 & 0 & 0 & 1\end{bmatrix}$

Find a basis of the image of the matrices in Exercises 27 through 33.

27. $\begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix}$ **28.** $\begin{bmatrix}1 & 1 & 1\\1 & 2 & 5\\1 & 3 & 7\end{bmatrix}$ **29.** $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$

30. $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ **31.** $\begin{bmatrix}1 & 5\\2 & 6\\3 & 7\\5 & 8\end{bmatrix}$

32. $\begin{bmatrix}0 & 1 & 2 & 0 & 0 & 3\\0 & 0 & 0 & 1 & 0 & 4\\0 & 0 & 0 & 0 & 1 & 5\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$

33. $\begin{bmatrix}0 & 1 & 2 & 0 & 3 & 0\\0 & 0 & 0 & 1 & 4 & 0\\0 & 0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$

34. Consider the 5×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

We are told that the vector $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ is in the kernel of A.

Write $\vec{v}_4$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

35. Show that there is a nontrivial relation among the vectors $\vec{v}_1, \dots, \vec{v}_m$ if (and only if) at least one of the vectors $\vec{v}_i$ is a linear combination of the other vectors $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$.

36. Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly dependent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$ necessarily linearly dependent? How can you tell?

37. Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$ necessarily linearly independent? How can you tell?

38. a. Let V be a subspace of $\mathbb{R}^n$. Let m be the largest number of linearly independent vectors we can find in V. (Note that $m \le n$, by Theorem 3.2.8.) Choose linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in V. Show that the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ span V and are therefore a basis of V. This exercise shows that any subspace of $\mathbb{R}^n$ has a basis.

If you are puzzled, think first about the special case when V is a plane in $\mathbb{R}^3$. What is m in this case?

b. Show that any subspace V of $\mathbb{R}^n$ can be represented as the image of a matrix.

39. Consider some linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$ and a vector $\vec{v}$ in $\mathbb{R}^n$ that is not contained in the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$. Are the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m, \vec{v}$ necessarily linearly independent? Justify your answer.

40. Consider an $n \times p$ matrix A and a $p \times m$ matrix B. We are told that the columns of A and the columns of B are linearly independent. Are the columns of the product AB linearly independent as well? *Hint*: Exercise 3.1.51 is useful.

41. Consider an $m \times n$ matrix A and an $n \times m$ matrix B (with $n \neq m$) such that $AB = I_m$. (We say that A is a *left inverse* of B.) Are the columns of B linearly independent? What about the columns of A?

42. Consider some perpendicular unit vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Show that these vectors are necessarily linearly independent. (*Hint*: Form the dot product of $\vec{v}_i$ and both sides of the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}.)$$

43. Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$. Are the vectors $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ linearly independent as well? How can you tell?

44. Consider linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$, and let A be an invertible $m \times m$ matrix. Are the columns of the following matrix linearly independent?

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix} A$$

45. Are the columns of an invertible matrix linearly independent?