# What is Distance? Problem Set 1

Mathcamp 2004, Week 2

due Wednesday

These problems are all of varying difficulty. Some of them (as many as I could make) don't require much work but do require some thinking and attention to details. Questions are, of course, welcome!

Homework is not required, but is recommended. The problems are fun and worth thinking about, and should not take up too much time. Nonetheless, you'll understand a lot more if you do them. I promise a quick response time on any homework turned in by the due date.

For your information, here are the most important definitions:

**Definition 1.** *A metric space is a set $X$ together with a function $d : X \times X \to [0, \infty)$ (called "the metric"; note how this definition nicely handles the positivity requirement) such that, for all $x, y, z \in X$:*

1. $d(x, y) = 0$ *if and only if* $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

**Definition 2.** *A set $B \subset X$ is an open ball if it is of the form $B(x, r) = \{y | y \in X, d(x, y) < r\}$ for some $x$ and $r$.*

**Definition 3.** *A set $U$ is open in a metric space if, for each $x \in U$, there is an open ball $B$ such that $x \in B$ and $B \subset U$.*

**Definition 4.** *An isometry is a bijective function $f : X \to Y$ such that, for all $x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$.*

Now, here are the problems!

1. The *ultra-triangle inequality* goes as follows: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. Note that this implies the standard triangle inequality. We say that a metric space is an ultrametric space if it satisfies this property.

   Recall that $E = \{0, 1\}$ and $E^\omega$ is the set of all infinite strings composed entirely of 0s and 1s. If $x$ and $y$ are two strings, let $\alpha$ be such that $x = \alpha x'$ and $y = \alpha y'$, and $x'$ and $y'$ start with different characters (thus, $\alpha$ is all of their overlap in the first few characters). Let $k$ be the length of $\alpha$. Then we define the distance $d(x, y) = \left(\frac{1}{2}\right)^k$.

   Prove that $E^\omega$ is an ultrametric space.

2. Let $(X, d)$ be an ultrametric space. Prove that:

   (a) Every triangle is isosceles: if $x, y, z \in X$, then at least two of $d(x, y), d(y, z), d(x, z)$ are equal.

   (b) Define the diameter of a set to be the maximum distance between two elements in that set. Show that an open ball of radius $r$ has diameter at most $r$.

   (c) Every point of a ball is a center of that ball; that is, if $y \in B(x, r)$, then $B(y, r) = B(x, r)$.

3. What happens if you get rid of the requirement that, for $x \neq y$, $d(x, y) > 0$? List as many consequences as possible. This is called a *pseudometric space.*

4. In an *anti-metric*, we change the triangle inequality to read instead that $d(x, z) \geq d(x, y) + d(y, z)$. Show that all anti-metric spaces are boring.