# What is Distance? Problem Set 2

Mathcamp 2004, Week 2

due Saturday

1. Which of the following spaces are complete? If they're not, give some intuition for what the completion might be (what are the points that are "missing").
   (a) The space $[0, 1]$ with the Euclidean metric.
   (b) The space $(0, 1)$ with the Euclidean metric.
   (c) $[0, 2]$ with the metric such that $d(a, b) = |a - b|$ if $a$ and $b$ are both rational, and $d(a, b) = 1$ if either $a$ or $b$ are irrational.
   (d) The space $\mathbb{R} \times \mathbb{R}$, where the distance between two points $a = (x_1, y_1)$ and $b = (x_2, y_2)$ is given by:
   $$d(a, b) = \begin{cases} |x_1 - x_2| & \text{if } y_1 = y_2 \\ 1 & \text{otherwise} \end{cases}$$
2. Recall that the discrete metric on any set $X$ is defined with $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ otherwise. Which discrete metric spaces are complete?
3. Show that $\mathbb{R}^2$ is complete under the Euclidean metric. What about under the taxicab metric?
4. In class, we tried (and failed) to define when a sequence "should" convege as follows:

   **Definition 1.** *A sequence "should" converge if, for any $\epsilon > 0$, there is an $N$ such that, for $n \geq N$, $d(x_n, x_{n+1}) < \epsilon$.*

   In other words, that $d(x_n, x_{n+1}) \to 0$. While this doesn't work for any metric space, show that if $X$ is an ultrametric space, a sequence $\{x_i\}$ is a Cauchy sequence if and only if it satisfies this weaker definition.
5. Show that $E^\omega$ is complete under the metric $d_{\frac{1}{2}}$.
6. Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces with an isometry $f$ between them.

(a) Let $U \subset X$, $V \subset Y$, both open. Show that $f(X)$ is open and that $f^{-1}(Y)$ is open.

(b) Show that, if a sequence $\{x_i\}$ converges in $X$, then $\{f(x_i)\}$ converges in $Y$.

(c) Show that $X$ is complete if and only if $Y$ is complete.