

# Proof Techniques Problem Set 1

Mathcamp 2004, Week 1

due Wednesday

Most of these homework problems shouldn't be hard but should still be cool. Do as many of them as you can; focus especially on the second section (the first section is short and fun, though).

Problems that have one star involve some kind of special creative idea; problems that have two stars are extra-hard.

## 1 Direct Proofs

1. Explain why the following diagram proves the Pythagorean Theorem.

2. Geometric series: prove that, for any  $n$ ,

$$\begin{aligned} a + ar + ar^2 + ar^3 + \dots + ar^n &= \frac{ar^{n+1} - a}{r - 1} \text{ if } r \neq 1 \\ &= (n + 1)a \text{ if } r = 1 \end{aligned}$$

(hint: let  $S = a + ar + ar^2 + \dots + ar^n$ ; compute  $rS - S$ )

## 2 Proofs by Contradiction

1. One way to define even numbers is to say that  $a$  is even if there exists an integer  $b$  such that  $a = 2b$ . Using this definition, a proof that a number

is even is trivial: one can prove 30 is even just by writing  $30 = 2 \times 15$ . Proving that a number is odd is also not hard; one can use the statement that “if  $k$  is even, then  $k + 1$  is not even” (since, if a number is divisible by 2, adding one means that it has remainder 1 when divided by 2). Use this to prove that 29 is not even.

2. Prove that there are no positive integers  $x$  and  $y$  such that  $x^2 - y^2 = 1$ .
3. Prove that there do not exist natural numbers  $x$  and  $y$  such that  $x^2 - 4y = 3$  (hint: consider two cases).
4. If  $p$  is a nonzero rational and  $q$  is irrational, prove that  $pq$  is irrational.
5. (a) Prove that  $\sqrt{3}$  is irrational.  
 (b) Prove that  $\sqrt[3]{2}$  is irrational.  
 (c) For which  $m$  and  $n$  must  $\sqrt[m]{n}$  be irrational?
6. Can there be a triangle such that the sum of any two angles is less than  $\frac{2\pi}{3}$ ?
7. (a) If  $a$  and  $b$  are positive integers such that  $\sqrt{a}$  and  $\sqrt{b}$  are both irrational, must  $\sqrt{a} - \sqrt{b}$  be irrational?  
 (b) (\*) Must  $\sqrt{a} + \sqrt{b}$  be irrational? (hint: the result of problem 4 may be helpful.)
8. (\*\*) In the game of Double Chess, each player gets two turns in a row. Prove that the first player has a non-losing strategy (hint: think about Chomp!).