Proof Techniques Problem Set 2

Mathcamp 2004, Week 1

due Thursday

Please give thought to all of these problems, and write up solutions for us to look over. Write at least one solution very carefully, so that we can comment on your proof-writing technique as well as the solutions themselves.

Note that we reserve the right to throw in questions requiring proof techniques you've learned before!

1. Every odd number (and thus prime number) can be written in the form $4k + 1$ or $4k + 3$ (this means that its remainder, when divided by 4, is either 1 or 3). We're going to work up to some variations on the infinitude of primes.

   (a) What kind of number do you get when multiplying two numbers of the form $4k + 1$? What about $4k + 3$? What about multiplying the two together?

   (b) Prove that there are infinitely many primes of the form $4k + 3$ (hint: suppose there were a finite list with $p_1 = 3$, $p_2 = 7$, $p_3 = 11$, etc. Consider $4p_2p_3p_4 \ldots p_n + 3$. Why do we omit p_1?).

   (c) Can you do the same thing for primes of the form $4k + 1$?

2. Suppose you are given 25 points in the plane such that, among any three of them, two are at distance less than one unit from each other. Prove that there is a circle of radius 1 that contains 13 of the points.

3. We showed in class that, if you choose 55 numbers between 1 and 100, there are two that differ by 9. Prove that there are also two that differ by 10, 12, and 13. Is it also true for 11? Why or why not?

4. Prove that it is impossible to color the plane with two colors (say, red and blue) such that each pair of points one unit apart has a different color.

5. Suppose that each point in the plane is colored either red or blue (without the conditions from the previous problem). Show that there is some rectangle all of whose vertices are the same color (hint: the trick with the pigeonhole principle is to figure out where to put your holes; once you get that, the rest falls into place nicely).

6. A point in the plane is a *lattice point* if both of its coordinates are integers (for example, $(0,1),(4,5),(-5,18)$ are all lattice points; $(\frac{1}{2},8)$ is not).
    (a) Prove that, if one chooses five lattice points in the plane, the midpoint of some two of them is a lattice point.
    (b) Do the same for nine lattice points in 3-dimensional space.
7. (a) (*) Suppose we choose five integers $a_1,\ldots,a_5$. Prove that either one of the a_i's is divisible by 5, or the sum of several numbers in a row is divisible by 5.
    (b) Prove the same result for arbitrary n: that is, given n integers $a_1,\ldots,a_n$, some sum of consecutive a_i's is divisible by n.
8. (*) Prove that, for any n, there is a number composed entirely of 1's and 0's that is divisible by n (hint: use a result we had from class).

Additional questions

You don't need to turn these in, but, after a while, these proofs get rather fun to churn out...

1. Given 37 numbers, show that you can choose seven of them such that their sum is divisible by 7 (this is different from previous problems because you must choose exactly seven numbers). Can you generalize?
2. Prove that, out of any set of 27 odd numbers all less than 100, some pair sum to 102.
3. Let x be any real number. Show that, among the numbers $x, 2x, 3x, \ldots, (n-1)x$, there is one that differs from an integer by at most $\frac{1}{n}$.