Math 481 Introduction to Differential Geometry

Assignment 1. Due Tuesday Feb. 3

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1. (A smooth atlas in the real projective plane)

   Let $\mathbb{R}P^2$ be the space of lines through the origin in $\mathbb{R}^3$. Let

   $$\{(U_i, \phi_i)\}_{i=1,2,3}$$

   be the collection of charts defined in class. For example, U_1 is the set of lines not contained in the x_2x_3-plane and $\phi_1(x_1, x_2, x_3) = (\frac{x_2}{x_1}, \frac{x_3}{x_1})$ where (x_1, x_2, x_3) is any point on the given line.

   (a) Show that every line through the origin in $\mathbb{R}^3$ is contained in either U_1, U_2 or U_3.

   (b) Show that $\phi_1(U_1)$ and $\phi_1(U_1 \cap U_2)$ are both open in $\mathbb{R}^2$.

   (c) Show that $\phi_2 \circ (\phi_1)^{-1}$ is smooth on $\phi_1(U_1 \cap U_2)$.

   (d) Show that

   $$\{(U_i, \phi_i)\}_{i=1,2,3}$$

   is a smooth atlas on $\mathbb{R}P^2$.

2. (A smooth atlas on S^2 with two charts.)

   Recall that the two-dimensional sphere can be realized as follows

   $$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

   Let

   $$U_1 = \{(x_1, x_2, x_3) \in S^2 \mid (x_1, x_2, x_3) \neq (1, 0, 0)\},$$
   $$U_2 = \{(x_1, x_2, x_3) \in S^2 \mid (x_1, x_2, x_3) \neq (-1, 0, 0)\}$$

   and consider the maps $\phi_1 \colon U_1 \to \mathbb{R}^2$ and $\phi_2 \colon U_2 \to \mathbb{R}^2$ defined by

   $$\phi_1(x_1, x_2, x_3) = \left(\frac{x_2}{1 - x_1}, \frac{x_3}{1 - x_1}\right)$$

   and

   $$\phi_1(x_1, x_2, x_3) = \left(\frac{x_2}{1 + x_1}, \frac{x_3}{1 + x_1}\right)$$

   Show that these charts determine a smooth atlas on S^2.