Math 481 Introduction to Differential Geometry

Assignment 5, Due Thursday March 19

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1. Let $f\colon S^1 \to \mathbb{R}$ be a smooth function on the circle. In this question you will show that the 1-form df vanishes at at least one point in S^1. In other words, there is a point q in S^1 such that $(df)_q\colon T_qS^1 \to \mathbb{R}$ takes every tangent vector $V \in T_qS^1$ to zero. To prove this we will use the following

   FACT: Every function $f\colon S^1 \to \mathbb{R}$ attains its maximum value at some point in S^1. In other there is a point $q \in S^1$ such that $f(q) \geq f(p)$ for all points $p \in S^1$.

   Let q be a point at which f takes its maximum value and let V be any element of T_qS^1. Prove that $(df)_q(V) = 0$.

   Hint: Let $\gamma\colon \mathbb{R} \to S^1$ be a curve such that $\gamma(0) = q$ and $\gamma_*(1) = V$. Recall that
   $$(df)_q(V) = \frac{d}{dt}_{t=0}(f(\gamma(t))).$$
   Prove that this is zero by considering the behavior of the function of one variable $f(\gamma(t))$ near $t = 0$.

2. In class, we defined a 1-form α on S^1 by defining its coordinate representatives for the usual charts $\{(U_i, \phi_i)\}_{i=1,\dots 4}$ as follows:
   $$\alpha^{U_1}(x) = -\frac{dx}{\sqrt{1-x^2}} \qquad \alpha^{U_2}(x) = \frac{dx}{\sqrt{1-x^2}}$$
   $$\alpha^{U_3}(y) = \frac{dy}{\sqrt{1-y^2}} \qquad \alpha^{U_4}(y) = -\frac{dy}{\sqrt{1-y^2}}.$$
   (a) Show that α is indeed a 1-form by checking that
   $$\alpha^{U_i} = \alpha^{U_j} \circ (\phi_j \circ \phi_i)$$
   for all relevant $i \neq j$. (We already did this for $i = 3$ and $j = 1$ in class.)

   (b) Show that α never vanishes. In other words, show that none of the maps $\alpha(p)\colon T_pS^1 \to \mathbb{R}$ take every tangent vector to zero.

(c) Conclude from (1) and (2b) that α is not equal to df for any function $f\colon S^1 \to \mathbb{R}$.

3. Let E be a 2-dimensional vector space with basis $\{e_1, e_2\}$. Let $\{\sigma^1, \sigma^2\}$ be the dual basis of E^*. Show that the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-tensors

$$e_1 \otimes \sigma^1, \quad e_1 \otimes \sigma^2, \quad e_2 \otimes \sigma^1 \text{ and } \quad e_2 \otimes \sigma^2$$

are linearly independent.

4. Consider the charts (U, ϕ) and (V, ψ) on $\mathbb{R}P^2$ where

$$U = \{[x : y : z] \in \mathbb{R}P^2 \mid z \neq 0\}, \quad \phi([x : y : z]) = \left(\frac{x}{z}, \frac{y}{z}\right) = (u_1, u_2)$$

and

$$V = \{[x : y : z] \in \mathbb{R}P^2 \mid y \neq 0\}, \quad \psi([x : y : z]) = \left(\frac{x}{y}, \frac{z}{y}\right) = (w_1, w_2).$$

Let A be a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-tensor field on $\mathbb{R}P^2$ such that in the chart (U, ϕ) we have

$$A^U(u_1, u_2) = ((u_1)^2 u_2)\frac{\partial}{\partial u_1} \otimes du_2.$$

Compute A^V on $U \cap V$.