

Math 481 Introduction to Differential Geometry

Assignment 6, Due Tuesday April 7

1. (a) Consider the 1-forms dx_1 and dx_2 on \mathbb{R}^3 . Using only the definition of the wedge product, compute

$$dx_1 \wedge dx_2(V_1, V_2)$$

where V_1 and V_2 are both vector fields on \mathbb{R}^3 .

- (b) Conclude from 1(a) that $dx_1 \wedge dx_2 = dx_1 \otimes dx_2 - dx_2 \otimes dx_1$.

2. The wedge product is associative in the sense that

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$

for any three differential forms α , β , and γ on a manifold M . In this question we will verify this fact for a particular example. Set $M = \mathbb{R}^3$ and let $\alpha = dx_1$, $\beta = dx_2$, and $\gamma = dx_3$. Using only the definition of the wedge product, show that

$$(dx_1 \wedge dx_2) \wedge dx_3(V_1, V_2, V_3) = dx_1 \wedge (dx_2 \wedge dx_3)(V_1, V_2, V_3)$$

where V_1 , V_2 , and V_3 are vector fields on \mathbb{R}^3 .

3. By the associativity of the wedge product described in question 2, the notation $\alpha_{i_1} \wedge \dots \wedge \alpha_{i_r}$ makes sense.

FACT: The r -forms $dx_{i_1} \wedge \dots \wedge dx_{i_r}$ with $i_1 < i_2 < \dots < i_r$ form a basis of $\Lambda^r(\mathbb{R}^n)$.

Use this fact to show that for an r -form α and an s -form β on a manifold M we have

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha.$$

Hint: prove it first for forms in \mathbb{R}^n .

4. **EXTRA PROBLEM** (for those students taking the class for extra credit). Prove the **FACT** stated in question 3.

5. Let α be a 1-form on a manifold M , and let (U, ϕ) and (V, ψ) be two charts on M with $U \cap V \neq \emptyset$. For the coordinate representative $\alpha^U(x) = \sum_i a_i(x) dx_i$ set

$$d(\alpha^U)(x) = \sum_i (da_i(x)) \wedge dx_i = \sum_{i,j} \frac{\partial a_i}{\partial x_j}(x) dx_j \wedge dx_i.$$

Define $d(\alpha^V)(y)$ in the analogous way. Show that

$$d(\alpha^V)(y) = (\phi \circ \psi^{-1})^*(d(\alpha^U)(x)).$$

(NOTE: This has been corrected from the previous version.)

Define $d\alpha$ by choosing its representative for a chart (U, ϕ) to be $(d\alpha)^U = d(\alpha^U)$. Conclude that $d\alpha$ is a 2-form on M .