Math 481 Introduction to Differential Geometry

Assignment 8, Due Thursday April 16

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1. Let ω be the 2-form on $\mathbb{R}^3$ defined as $\omega = dx \wedge dy + dy \wedge dz$. Consider a 2-cube $\gamma\colon [0,1]^2 \to \mathbb{R}^3$ given by

$$\gamma(t,s) = (\cos 2\pi t \cos 2\pi s, \sin 2\pi t \cos 2\pi s, \sin 2\pi s).$$

   Compute $\int_\gamma \omega$.

2. For $n \geq r \geq 1$ we define the map $I\colon \Lambda^r(\mathbb{R}^n) \to \Lambda^{r-1}(\mathbb{R}^n)$ as follows. For an r-form

$$\omega(x) = \sum_{i_1 < \ldots < i_r} f_{i_1 \ldots i_r}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_r}$$

   we set

$$(I(\omega))(x) = \sum_{i_1<\ldots<i_r} \sum_{j=1}^{r} \left(\int_0^1 (-1)^{j+1} t^{r-1} f_{i_1\ldots i_r}(tx)\, dt \right) x_{i_j}\, dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_r}.$$

   (a) Prove that $\omega = I(d\omega) + d(I\omega)$ for every $\omega \in \Lambda^r(\mathbb{R}^n)$.

   (b) Conclude that every closed r-form ω on $\mathbb{R}^n$ (where $r \geq 1$) is exact. That is, show that if $d\omega = 0$ then there is some $(r-1)$-form α such that $\omega = d\alpha$.

3. Let $\gamma\colon [0,1]^r \to M$ be an r-cube where $r \geq 2$.

   (a) For $1 \leq i \leq j \leq r-1$ compute the face-maps $(\gamma_{(i,\alpha)})_{(j,\beta)}(x)$ and $(\gamma_{(j+1,\beta)})_{(i,\alpha)}(x)$ for $x \in [0,1]^{r-2}$. Conclude that $(\gamma_{(i,\alpha)})_{(j,\beta)} = (\gamma_{(j+1,\beta)})_{(i,\alpha)}$ for $1 \leq i \leq j \leq r-1$.

   (b) Prove that $\partial(\partial\gamma) = 0$.

4. (**EXTRA PROBLEM**) Let $M = \mathbb{R}^2 \setminus \{0\}$, the plane with the origin removed. For $R > 0$ and an integer n define the 1-cube $\gamma_{R,n}\colon [0,1] \to M$ by

$$\gamma_{R,n}(t) = (R\cos(2\pi n t), R\sin(2\pi n t)).$$

   (a) Show that for any integer n and any positive real numbers R_1, r_2 there is a 2-cube $c\colon [0,1]^2 \to M$ such that $\partial c = \gamma_{R_1,n} - \gamma_{R_2,n}$.

(b) Use Stokes's Theorem to show that if $R > 0$ and $n_1 \neq n_2$ that there is no 2-cube $c\colon [0,1]^2 \to M$ such that $\partial c = \gamma_{R,n_1} - \gamma_{R,n_2}$.

5. (**EXTRA PROBLEM**) An r-form ω is called *decomposable* if there are 1-forms $\alpha_1, \ldots \alpha_r$ such that $\omega = \alpha_1 \wedge \ldots \wedge \alpha_r$.

   (a) Show that the 2-form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ on $\mathbb{R}^4$ is **not** decomposable.

   (b) Show that every 2-form on $\mathbb{R}^3$ is decomposable.

   (c) Show that every $(n-1)$-form on $\mathbb{R}^n$ is decomposable.