Math 481 Introduction to Differential Geometry

Assignment 9, Due Thursday April 23

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1. Let ω be an r-form on an n-dimensional manifolds M such that there is an r-chain α in M with $\partial\alpha = 0$ and $\int_\alpha \omega \neq 0$. Show that ω can not be written as $d\sigma$ for any $(r-1)$-form σ.

2. Describe the two dimensional torus T^2 as the following subset of $\mathbb{R}^4$:

$$T^2 = \{(x, y, u, v) \in \mathbb{R}^4 \mid x^2 + y^2 = 1,\ u^2 + v^2 = 1\}.$$

   (a) Consider the following 1-cubes on T^2:

$$\gamma_1(t) = (\cos 2\pi t, \sin 2\pi t, 1, 0)$$

   and

$$\gamma_2(t) = (1, 0, \cos 2\pi t, \sin 2\pi t).$$

   Show that $\partial\gamma_1 = 0 = \partial\gamma_2$.

   (b) Consider the following 1-forms on T^2 defined (by restriction) as follows:

$$\omega_1 = \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

   and

$$\omega_2 = \frac{-v}{u^2 + v^2}du + \frac{u}{u^2 + v^2}dv.$$

   Compute $\int_{\gamma_1} \omega_1$ and $\int_{\gamma_2} \omega_2$. Conclude that neither ω_1 nor ω_2 is equal to df for some function f on T^2.

   (c) Consider the mapping $F\colon (0, 2\pi) \times (0, 2\pi) \to T^2$ defined by

$$F(\theta_1, \theta_2) = (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2).$$

   Show that $d(F^*\omega_1) = 0 = d(F^*\omega_2)$. Conclude that $d\omega_1 = 0 = d\omega_2$.

3. Consider the two dimensional sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

   Let ω be the 2-form on S^2 defined by restricting to S^2 the form

$$\alpha = x\,dy \wedge dz + y\,dx \wedge dz + z\,dx \wedge dz.$$

(a) Show that ω is nonvanishing. (Hint: for our standard charts (U_i, ϕ_i) consider the coordinate representatives $\omega^{U_i} = (\phi_i^{-1})^*\alpha$. It suffices to analyze just one of these representatives, say ω^{U_1}.)

(b) Compute $\int_V \omega$ where S^2 is oriented by the form ω, and V is the subset of S^2 defined by

$$V = \{(x, y, z) \in S^2 \mid |x| \leq 1/2, |y| \leq 1/2, z > 0\}.$$

You can leave your answer in the form of a definite double integral.