Math 481 Test 1 Solutions.

1) a) A vector field V is a smooth mapping $V: M \to TM$ such that $\pi(V(p)) = p$ for all $p \in M$.

b) $q \in N$ is a regular value of $F: M \to N$ if every point p in $F^{-1}(q)$ is a regular point of F

i.e. $F_* : T_pM \to T_qN$ is onto.

c) If $q \in N$ is a regular value of $F: M \to N$, then $F^{-1}(q)$ is a submanifold of M of dimension $m-n$.

d) $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is a smooth atlas for M if

i) $\bigcup_{\alpha \in A} U_\alpha = M$

ii) $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ is 1-1 for all $\alpha \in A$

iii) $\phi_\alpha(U_\alpha) \subset \mathbb{R}^m$ is open for all $\alpha \in A$

iv) $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^m$ is open for all $\alpha, \beta \in A$

v) $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^m$.

is smooth for all $U_\alpha \cap U_\beta \neq \emptyset$.

2 a) Need

$$V^{U_4}(y) = (\phi_4 \circ \phi_1^{-1})_* V^{U_1}(x)$$

or $$V^{U_1}(x) = (\phi_1 \circ \phi_4^{-1})_* V^{U_4}(y).$$

2b) $\phi_4 \circ \phi_1^{-1}(x) = \phi_4\left(x, \sqrt{1-x^2}\right)$

$= \sqrt{1-x^2}$

So $(\phi_4 \circ \phi_1^{-1})_* V^{u_1}(x)$

$= \frac{d}{dx}\left(\sqrt{1-x^2}\right)\left(-x\sqrt{1-x^2}\right)$

$= \frac{-x}{\sqrt{1-x^2}}\left(-x\sqrt{1-x^2}\right.$

$= x^2$

$= 1-y^2$

$= V^{u_4}(y).$

3) Define $F : \mathbb{R}^4 \to \mathbb{R}^2$ by.

$$F(x_1, x_2, x_3, x_4) = \left(x_1^2 + \frac{x_2^2}{2} + \frac{x_3^2}{3} + \frac{x_4^2}{4} \,,\; 4x_1 + 3x_2 + 2x_3 + x_4 \right)$$

Then $M = F^{-1}((1,0))$ and it suffices to show that $(1,0)$ is a regular value of F.

Let $x = (x_1, x_2, x_3, x_4)$ belong to $F^{-1}((1,0))$.

Then we have
$$\begin{cases} x_1^2 + \dfrac{x_2^2}{2} + \dfrac{x_3^2}{3} + \dfrac{x_4^2}{4} = 1 & (1) \\ 4x_1 + 3x_2 + 2x_3 + x_4 = 0 & (2) \end{cases}$$

The differential at x is given by

$$F_* v = \begin{bmatrix} 2x_1 & x_2 & \frac{2}{3}x_3 & \frac{1}{2}x_4 \\ 4 & 3 & 2 & 1 \end{bmatrix} v$$

Need to show that the matrix has rank $= 2$.

Note that rank ≥ 1 since second row is nonzero.

If rank $= 1$ then the rows are linearly dependent

ie. $[2x_1, x_2, \frac{2}{3}x_3, \frac{1}{2}x_4] = c\,[4, 3, 2, 1]$

for some $c \neq 0$.

But this implies that $x_1 = 2c$, $x_2 = 3c$, $x_3 = 3c$,
$x_4 = 2c$.

Then $4x_1 + 3x_2 + 2x_3 + x_4 = (8 + 9 + 6 + 2)c \neq 0$

So if rank $= 1$ then (x_1, x_2, x_3, x_4) can not satisfy equation (2) and so is not in $F^{-1}((1,0))$

Therefore $(1,0)$ is a regular value and M is a submanifold of $\mathbb{R}^4$ of dimension $4-2=2$.

4 a) $\phi_1^{-1}(x_2, x_3) = \left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3 \right)$.

So ϕ_1 is 1-1 on U_1

b) $f \circ \phi_1^{-1}(u, v) = f\left(\sqrt{1 - u^2 - v^2}, u, v \right)$

$= 1 - u^2 - v^2$.

c) $(df)^{U_1} = d(f \circ \phi_1^{-1})$

$= -2u\,du - 2v\,dv$.

d) $f_* : T_p S^1 \to T_{f(p)} \mathbb{R}$.

$V \mapsto (f \circ \phi_1^{-1})_* V^{U_1}$

Now $V^{U_1} = \left.\frac{d}{dt}\right|_{t=0} \left(\phi_1(\gamma(t)) \right)$

$= \left.\frac{d}{dt}\right|_{t=0} (0, \sin t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So $f_*(v) = (f \circ \phi_1^{-1})_* \, v^{u_1}$

$$= \begin{bmatrix} -2u & -2v \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 0$$

5 a) $\frac{dx}{dt} = V(x)$

$$\Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \frac{dx_1}{dt} = x_1 & (1) \\ \frac{dx_2}{dt} = 2 & (2) \end{cases}$$

(1): $\frac{dx_1}{dt} = x_1 \Leftrightarrow \frac{dx_1}{x_1} = dt$

$$\Leftrightarrow \ln|x_1| = t + C.$$

$$\Rightarrow \quad x_1 = A e^t$$

$$\Rightarrow \quad x_1(t) = x_1(0) e^t.$$

(2): $\dfrac{dx_2}{dt} = 2 \iff x_2(t) = 2t + x_2(0)$

So for $\gamma(0) = (1,2)$ we have.

$$\gamma(t) = (e^t, 2t+2)$$

~~b) $\phi_t(x_1, x_2) = (x e^t, 2t+$~~

b) $\phi_t(x_1, x_2) = (x_1 e^t, 2t + x_2)$

$$\phi_1(x_1, x_2) = (x_1 e, 2 + x_2)$$