Math 481 Hwk 1 Solutions

1) a) Assume there is a line L which is not contained in $U_1 \cup U_2 \cup U_3$.

Let (x_1, x_2, x_3) be a point on L which is different from $(0,0,0)$.

Since L is not in U_1 we have $x_1 = 0$.

Similarly $x_2 = 0$ and $x_3 = 0$.

This contradicts our choice of (x_1, x_2, x_3), so no such L exists.

b) $\phi_1(U_1) = \left\{ \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right) \in \mathbb{R}^2 \,\middle|\, x_1 \neq 0 \right\}$

$\supset \{ (x_2, x_3) \in \mathbb{R}^2 \}$

$= \mathbb{R}^2.$

this is clearly open.

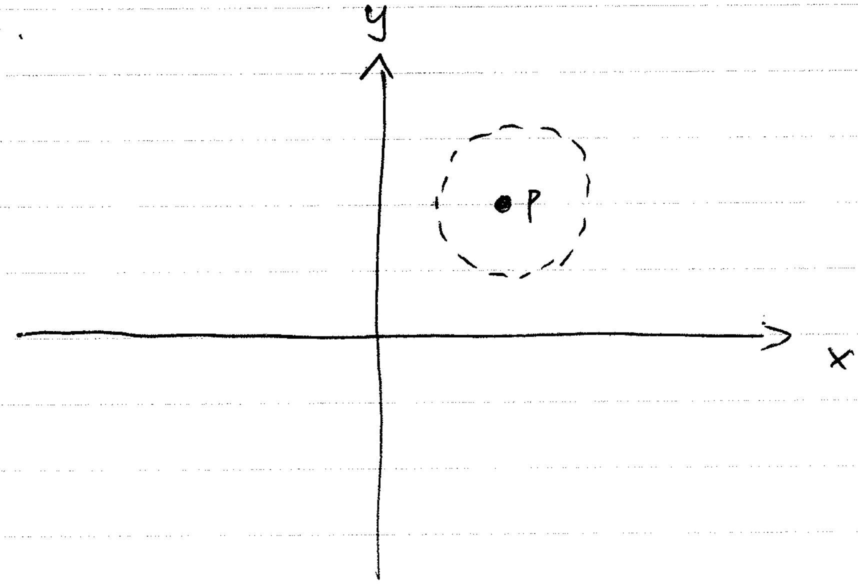
$\phi_1(U_1 \cap U_2)$

$= \left\{ \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right) \in \mathbb{R}^2 \,\middle|\, x_1 \neq 0, x_2 \neq 0 \right\}$

$= \mathbb{R}^2 \setminus$ y-axis

For a point $p = (a,b)$ in $\mathbb{R}^2 \setminus$ y-axis the ball of radius $a/2$ around p is contained in $\mathbb{R}^2 \setminus$ y-axis.

i.e.



So $\phi_1(U_1 \cap U_2)$ is open.

c) $\phi_2 \circ \phi_1^{-1} : \mathbb{R}^2 \setminus \text{y-axis} \to \mathbb{R}^2$.

$$\phi_2 \circ \phi_1^{-1}(u,v) = \phi_2(1, u, v) = \left(\frac{1}{u}, \frac{v}{u}\right)$$

This is clearly smooth away from $u=0$

(d)

Arguing as in parts (a) (b) and (c) it follows that $\{(U_i, \phi_i)\}_{i=1,2,3}$ is an atlas for $\mathbb{RP}^2$ if we can show that $\phi_i : U_i \to \mathbb{R}^2$ is one-to-one for $i=1,2,3$.

It suffices to show that these maps are invertible.

$\phi_1^{-1}(u,v) =$ the line through the origin and the point $(1,u,v)$

$\phi_2^{-1}(u,v) =$ ———— " ———— $(u,1,v)$

$\phi_3^{-1}(u,v) =$ ———— " ———— $(u,v,1)$

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2) (i) $U_1 \cup U_2 = S^2$ since $(1,0,0) \in U_2$

(ii) to prove that ϕ_1 is one-to-one we compute ϕ_1^{-1}

$$\phi_1(x_1, x_2, x_3) = \left(\frac{x_2}{1-x_1}, \frac{x_3}{1-x_1} \right) = (u, v).$$

$$u^2+v^2 = \frac{x_2^2 + x_3^2}{(1-x_1)^2} = \frac{1-x_1^2}{(1-x_1)^2} = \frac{1+x_1}{1-x_1}$$

$$\Rightarrow \quad x_1 = \frac{u^2+v^2-1}{u^2+v^2+1}.$$

Now $1-x_1 = 1 - \dfrac{u^2+v^2-1}{u^2+v^2+1} = \dfrac{2}{u^2+v^2+1}$

So $\dfrac{x_2}{1-x_1} = u \Rightarrow x_2 = \dfrac{2u}{u^2+v^2+1}$

and $\dfrac{x_3}{1-x_1} = v \Rightarrow x_3 = \dfrac{2v}{u^2+v^2+1}$

So $\phi_1^{-1}(u,v) = \left(\dfrac{u^2+v^2-1}{u^2+v^2+1}, \dfrac{2u}{u^2+v^2+1}, \dfrac{2v}{u^2+v^2+1} \right).$

A similar computation yields a formula for ϕ_2^{-1} and confirms that ϕ_2 is one-to-one.

(iii) $$\phi_1(U_1) = \left\{\left(\frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}\right) \,\middle|\, x_1 \neq 1\right\}$$

$$= \mathbb{R}^2.$$

$$\phi_1(U_1 \cap U_2) = \left\{\left(\frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}\right) \,\middle|\, x_1 \neq \pm 1\right\}$$

$$= \mathbb{R}^2 \setminus (0,0)$$

Both these sets are open.

iv) $$\phi_2 \circ \phi_1^{-1}(u,v) = \phi_2\left(\frac{u^2+v^2-1}{u^2+v^2+1}, \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}\right)$$

$$= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

This is smooth away from $(u,v) = (0,0)$.

Hence $\{(U_i, \phi_i)\}_{i=1,2}$ is a smooth atlas for S^2.