# Math 481 HWK 2 Solutions



1) Set $F : \mathbb{R}^{n+1} \to \mathbb{R}$

$$(x_1, \ldots, x_{n+1}) \mapsto x_1^2 + \ldots + x_{n+1}^2$$

Then $S^n = F^{-1}(1)$

Now $F_*(v) = \begin{bmatrix} 2x_1 & \cdots & 2x_{n+1} \end{bmatrix} v$

<u>Claim</u> $F_*$ is surjective iff $x \neq (0, \ldots, 0)$

<u>Pf</u> Suppose $x \neq (0, \ldots, 0)$

We may assume that $x_1 \neq 0$

Choose $v = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Then at $x$ we have.

$$F_*(v) = 2x_1 v_1 \in T_{F(x)}\mathbb{R}.$$

Clearly every element $w \in T_{F(x)}\mathbb{R}$ is in the image of $F_*$. ( choose $v_1 = \frac{w}{2x_1}$ )

From this we see that every point in $\mathbb{R}^{n+1}$ other than the origin is a regular point of $F$.

Since $F^{-1}(1)$ doesn't contain the origin, $1$ is a regular value of $F$.

By the RVT $S^n = F^{-1}(1)$ is then a submanifold of $\mathbb{R}^{n+1}$ and hence a manifold itself.

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27) Set $F: M(n) \to \mathrm{Sym}(n)$

$$A \mapsto AA^T$$

Then $O(n) = F^{-1}(\mathbb{I})$.

As in class, we have

$$F_*(V) = VA^T + AV^T$$

for the differential at $A \in M(n)$.

For $A \in F^{-1}(\mathbb{I})$ we want to verify that $F_*$ is onto.

For $W \in T_{F(A)} \mathrm{Sym}(n)$ we need to find a $V \in T_A M(n)$ such that:

$$F_*(V) = W$$

$$\Leftrightarrow \quad VA^T + AV^T = W \qquad (*)$$

Now $W \in T_{F(A)} \mathrm{Sym} \Rightarrow W = W^T$.

To solve ⊛ for $V$ it suffices to find $V$ s.t. $VA^T = \frac{1}{2}W$, since.

$$
\begin{aligned}
VA^T + AV^T &= VA^T + (VA^T)^T \\
&= \tfrac{1}{2}W + (\tfrac{1}{2}W)^T \\
&= \tfrac{1}{2}W + \tfrac{1}{2}W \\
&= W
\end{aligned}
$$

But $VA^T = \frac{1}{2}W \iff V = \frac{1}{2}WA$.

So $V = \frac{1}{2}WA$ satisfies ⊛ and hence $\mathbb{I}$ is a regular value of $F$.

$\therefore$ $F^{-1}(\mathbb{I}) = O(n)$ is a submanifold of $M(n)$.

The dimension of $O(n)$ is equal to

$\dim(M(n)) - \dim(\mathrm{Sym}(n))$

$$= n^2 - (1 + 2 + \cdots + n)$$

$$= n^2 - \frac{n(n+1)}{2}$$

$$= \frac{n^2}{2} - \frac{n}{2}$$

$$= \frac{n(n-1)}{2}.$$