Math 481 Hwk 3 Solutions

1 a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

So $F : \mathbb{R}^4 \to \mathbb{R}^3$

$$(a,b,c,d) \mapsto (a^2+b^2,\ ac+bd,\ c^2+d^2)$$

b) $$F_* = \begin{bmatrix} 2a & 2b & 0 & 0 \\ c & d & a & b \\ 0 & 0 & 2c & 2d \end{bmatrix}$$

c) $$F(A) = \mathbb{I} \iff \begin{cases} a^2+b^2 = 1 & (1) \\ ac+bd = 0 & (2) \\ c^2+d^2 = 1 & (3) \end{cases}$$

Denote the rows of F_* by r_1, r_2 and r_3.

i.e $F_* = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$.

Note ①, ② and ③ $\Rightarrow$ r_1, r_2 and r_3 are not rows of zeros.

$$r_1 \circ r_2 = 2ac + 2bd = 0 \quad \text{by } ②$$

$$r_1 \circ r_3 = 0$$

$$r_2 \cdot r_3 = 2ac + 2bd = 0 \quad \text{by } ②$$

Hence the rows of F_* are nonzero and orthogonal. It follows that the rank of F_* is 3. Therefore F_* is onto for all $A \in F^{-1}(\mathbb{I})$ and so $F^{-1}(\mathbb{I}) = O(2)$ is a submanifold of $\mathbb{R}^4$ of dimension 1.

2a) $\phi_{v,B}(p) = \phi_{v,B}(q)$

$\Leftrightarrow \quad B(\phi(p)) + v = B(\phi(q)) + v$

$\Leftrightarrow \quad B(\phi(p)) = B(\phi(q))$

$\Leftrightarrow \quad \phi(p) = \phi(q)$ (Since B is invertible)

$\Leftrightarrow \quad p = q$ (Since ϕ is one-to-one)

b) $\phi_{v,B}(p) = B(\phi(p)) + v$

$\Leftrightarrow \quad \phi(p) = B^{-1}(\phi_{v,B}(p) - v)$

$\Leftrightarrow \quad p = \phi^{-1}\left(B^{-1}(\phi_{v,B}(p) - v)\right)$

So $\phi_{v,B}^{-1}(x) = \phi^{-1}\left(B^{-1}(x - v)\right)$

Now $\psi \circ \phi_{v,B}^{-1}(x) = \psi \circ \phi^{-1}(B^{-1}(x-v))$

Since (U,ϕ) and (V,ψ) are compatible the overlap map $\psi \circ \phi^{-1}$ is smooth

The map $x \mapsto B^{-1}(x-v)$ is also smooth.

By the chain rule $\psi \circ \Phi^{-1}_{v,B}$ is smooth

Now we consider the other composition

$$\Phi^{-1}_{v,B} \circ \psi^{-1}(x) = B(\phi \circ \psi^{-1}(x)) + v$$

This is again smooth because $\phi \circ \psi^{-1}$ is smooth and $x \mapsto Bx + v$ is smooth.

c) $\phi_{v,B}(p) = B(\phi(p)) + v$

Set $v = -B(\phi(p))$. to get

$\phi_{v,B}(p) = 0$.