Math 481 Hwk 4 Solutions

1 b) $$\frac{dx}{dt}(t) = V(x(t))$$

$$\Leftrightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ (x_2(t))^2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \dot{x}_1(t) = 1 & \text{(1)} \\ \dot{x}_2(t) = (x_2(t))^2 & \text{(2)} \end{cases}$$

Need to solve this system of 1st order ODE's with initial condition $(x_1(0), x_2(0)) = (a, b)$

Equation (1) has the solution

$$x_1(t) = t + a$$

If $b = 0$, then (2) has the solution

$$x_2(t) = 0$$

Otherwise (2) is separable and we get

$$\frac{dx_2}{(x_2)^2} = dt$$

$$-\frac{1}{x_2} = t + c$$

$$x_2(t) = \frac{-1}{t+c}$$

Since $x_2(0) = b \neq 0$ we have $c = -\frac{1}{b}$ and

$$x_2(t) = \frac{-1}{t - \frac{1}{b}}.$$

The integral curve $\gamma(t)$ with $\gamma(0) = (a, b)$ is then

$$\gamma(t) = \begin{cases} (t + a, b) & \text{if } b = 0 \\ \left(t + a, \frac{-1}{t - \frac{1}{b}}\right) & \text{if } b \neq 0 \end{cases}$$

1c) If $b=0$ then $\gamma(t)$ is defined for all $t \in \mathbb{R}$.

If $b<0$ then $\gamma(t)$ is defined for all $t \in (\frac{1}{b}, \infty)$.

If $b>0$ then $\gamma(t)$ is defined for all $t \in (-\infty, \frac{1}{b})$.

1d) By 1c) $\gamma(1)$ is defined if $b<1$.

So for $x=(a,b)$, $\phi_1(x)$ exists if $b<1$.

2a) <u>Claim</u>: $T(M \times N) = TM \times TN$.

Let V be the vector field on S^1 defined as the restriction of the vector field $V(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$.

Note V is nonvanishing on S^1.

Let W be any vector field on S^2.

<u>Example</u> $W^u = 0$ for all charts (u, ϕ) of S^2.

Define the vector field $V \times W$ on $S^1 \times S^2$ by setting

$$V \times W : S^1 \times S^2 \longrightarrow T(S^1 \times S^2) = TS^1 \times TS^2$$
$$(p, q) \mapsto (V(p), W(q))$$

Then $v \times w$ is nonvanishing.

2b)