Math 481 Hwk 5 Solutions

1) $f: S^1 \to \mathbb{R}$ attains its maximum at q.

$\Rightarrow$ $f(\gamma(t))$ attains its maximum at $t=0$ (since $\gamma(0)=q$).

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = 0.$$

$\Rightarrow$ $(df)_q(V) = 0$ since $\gamma_*(1) = V$.

2) a) $i=2$ $j=4$

$$\phi_4 \circ \phi_2^{-1}(x) = \phi_4\left(x, -\sqrt{1-x^2}\right) = -\sqrt{1-x^2}$$

$$(\phi_4 \circ \phi_2^{-1})_*\left(\partial/\partial x\right) = \frac{x}{\sqrt{1-x^2}}\,\partial/\partial y.$$

So $\alpha^{U_4}(y) \circ (\phi_4 \circ \phi_2^{-1})_* \left(a\,\partial/\partial x\right)$

$$= -\frac{dy}{\sqrt{1-y^2}}\left(\frac{a\,x}{\sqrt{1-x^2}}\frac{\partial}{\partial y}\right)$$

$$= -a\,\frac{x}{\sqrt{1-y^2}\,\sqrt{1-x^2}}$$

$$= \frac{+a}{\sqrt{1-x^2}}$$

and $\alpha^{U_2}(x)\left(a\,\partial/\partial x\right) = \frac{dx}{\sqrt{1-x^2}}\left(a\,\partial/\partial x\right)$

$$= \frac{a}{\sqrt{1-x^2}} \checkmark$$

etc.

2b) $\alpha(p) = 0 \iff \alpha^{U_i}(\phi_i(p)) = 0$ for $p \in U_i$

But none of the local representatives α^{U_i} vanish on their domains.

So $\alpha(p) \neq 0$ for all $p \in S^1$.

If there is a function $f: S^1 \to \mathbb{R}$ such that $\alpha = df$ then by Q1 α must vanish at some point in S^1. But we just proved that α never vanishes.

3) Consider a linear combination

$$T = \sum_{i,j} T^i_j \, e_i \otimes \sigma^j \quad \text{such that} \quad T = 0.$$

We must prove that $T^i_j = 0$ for $i,j = 1,2$.

$$T = 0 \quad \text{in} \quad \mathcal{T}_{(1,1)}(E)$$

$$\Leftrightarrow \quad T(\sigma^k, e_\ell) = 0 \quad \text{for all } k,\ell = 1,2$$

$$\Leftrightarrow \quad T^k_\ell = 0 \quad \text{for all } k,\ell = 1,2.$$

which is what we wanted.

4) $A^u = u_1^2 u_2 \ \partial/\partial u_1 \otimes du_2$.

$$(A^u)^1_2 = u_1^2 u_2$$

$$(A^u)^1_1 = (A^u)^2_1 = (A^u)^2_2 = 0$$

$$\psi \circ \phi^{-1}(u_1, u_2) = \psi([u_1 : u_2 : 1]) = \left(\frac{u_1}{u_2}, \frac{1}{u_2}\right)$$

$$\phi \circ \psi^{-1}(w_1, w_2) = \phi([w_1 : 1 : w_2]) = \left(\frac{w_1}{w_2}, \frac{1}{w_2}\right)$$

$$(A^v)^1_1 = (A^u)^1_2 \frac{\partial w_1}{\partial u_1} \frac{\partial u_2}{\partial w_1} = u_1^2 u_2 \left(\frac{1}{u_2}\right) \cdot 0 = 0$$

$$(A^v)^1_2 = (A^u)^1_2 \frac{\partial w_1}{\partial u_1} \frac{\partial u_2}{\partial w_2} = u_1^2 u_2 \left(\frac{1}{u_2}\right)\left(-\frac{1}{w_2^2}\right)$$

$$= \frac{-u_1^2}{w_2^2}$$

$$= \frac{-w_1^2}{w_2^4}$$

$$\cancel{(A^v)^2_1 = (A^u)^1_2 \frac{\partial w_1}{\partial u_2} \frac{\partial u_1}{\partial w_2} = u_1^2 u_2 \left(-\frac{u_1}{u_2^2}\right)\left(-\frac{w_1}{w_2^2}\right)}$$

$$(A^v)^2_1 = (A^u)^1_2 \frac{\partial w_2}{\partial u_1} \cdot \frac{\partial u_2}{\partial w_1} = 0$$

$$(A^v)^2_2 = (A^u)^1_2 \frac{\partial w_2}{\partial u_1} \frac{\partial u_2}{\partial w_2} = 0$$

$$A^v(w_1, w_2) = -\frac{w_1^2}{w_2^4} \frac{\partial}{\partial w_1} \otimes dw_2.$$