Math 481 Hwk 6 Solutions

1a) $$\alpha \wedge \beta (V_I) = \sum_{J K} \delta^{J K}_{I} \alpha(V_J)\, \beta(V_K)$$

$I = (1\ 2)$

$J \in \{ (1), (2) \}$

$K \in \{ (2), (1) \}$

$$dx_1 \wedge dx_2 (V_1, V_2) = \delta^{(12)}_{(12)} dx_1(V_1)\, dx_2(V_2) + \delta^{(21)}_{(12)} dx_1(V_2)\, dx_2(V_1)$$

$$= dx_1(V_1)\, dx_2(V_2) - dx_1(V_2)\, dx_2(V_1)$$

1b) $(dx_1 \otimes dx_2 - dx_2 \otimes dx_1)(V_1, V_2)$

$= dx_1(V_1)\, dx_2(V_2) - dx_2(V_1)\, dx_1(V_2)$

$= dx_1 \wedge dx_2 (V_1, V_2)$

$\checkmark$

2) Let's first compute $\alpha \wedge \beta \, (V_I)$ where

$\alpha \in \Lambda^2(M)$ $\quad \beta \in \Lambda^1(M)$ and $I = (1,2,3)$

$\underline{J} \in \{ (12), (13), (2,3) \}$

$\underline{K} \in \{ (3), (2), (1) \}$

$$\alpha \wedge \beta \, (V_I) = \delta^{(123)}_{(123)} \alpha(V_1, V_2)\beta(V_3) + \delta^{(132)}_{(123)} \alpha(V_1, V_3)\beta(V_2)$$

$$+ \delta^{(231)}_{(123)} \alpha(V_2, V_3)\beta(V_1)$$

$$= \alpha(V_1, V_2)\beta(V_3) - \alpha(V_1, V_3)\beta(V_2)$$

$$+ \alpha(V_2, V_3)\beta(V_1)$$

So $(dx_1 \wedge dx_2) \wedge dx_3 \, (V_1, V_2, V_3)$

$$= (dx_1 \wedge dx_2)(V_1, V_2) \, dx_3(V_3) - (dx_1 \wedge dx_2)(V_1, V_3) \, dx_3(V_2)$$

$$+ (dx_1 \wedge dx_2)(V_2, V_3) \, dx_3(V_1)$$

$$= \left(dx_1(v_1)\, dx_2(v_2) - dx_1(v_2)\, dx_2(v_1) \right) dx_3(v_3)$$

$$- \left(dx_1(v_1)\, dx_2(v_3) - dx_1(v_3)\, dx_2(v_1) \right) dx_3(v_2)$$

$$+ \left(dx_1(v_2)\, dx_2(v_3) - dx_1(v_3)\, dx_2(v_2) \right) dx_3(v_1)$$

Now $\quad dx_1 \wedge (dx_2 \wedge dx_3)\,(v_1, v_2, v_3)$

$$= -(dx_2 \wedge dx_3) \wedge dx_1 \,(v_1, v_2, v_3)$$

$$= (dx_3 \wedge dx_2) \wedge dx_1 \,(v_1, v_2, v_3)$$

$$= (dx_3 \wedge dx_2)(v_1, v_2)\, dx_1(v_3) - (dx_3 \wedge dx_2)(v_1, v_3)\, dx_1(v_2)$$

$$+ (dx_3 \wedge dx_2)(v_2, v_3)\, dx_1(v_1)$$

$$= \left(dx_3(v_1)\, dx_2(v_2) - dx_3(v_2)\, dx_2(v_1) \right) dx_1(v_3)$$

$$- \left(dx_3(v_1)\, dx_2(v_3) - dx_3(v_3)\, dx_2(v_1) \right) dx_1(v_2)$$

$$+ \left(dx_3(v_2)\, dx_2(v_3) - dx_3(v_3)\, dx_2(v_2) \right) dx_1(v_1)$$

Comparing terms in both expressions we get the desired equality.

<u>Note</u> $(dx_1 \wedge dx_2) \wedge dx_3 \, (V_1, V_2, V_3)$

$$= dx_1 \wedge (dx_2 \wedge dx_3)(V_1, V_2, V_3)$$

$$= \det \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix}$$

3×3 matrix with column vectors V_1, V_2, and V_3.

3) Let (U, ϕ) be a chart in M. To prove that $\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha$ it suffices to show that $\alpha^u \wedge \beta^u = (-1)^{rs} \beta^u \wedge \alpha^u$.

In fact, since $\alpha \wedge (a\beta + b\gamma) = a\, \alpha \wedge \beta + b\, \alpha \wedge \gamma$ we can prove this for α^u and β^u of the form $\alpha^u(x) = f(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_r}$

and $\beta^{u}(x) = g(x)\, dx_{j_1} \wedge \dots \wedge dx_{j_s}$

$$\alpha^{u}(x) \wedge \beta^{u}(x)$$

$$= \left(f(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_r}\right) \wedge \left(g(x)\, dx_{j_1} \wedge \dots \wedge dx_{j_s}\right)$$

$$= f \cdot g\,(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_r} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_s}$$

$$= (-1)^{r} f \cdot g\,(x)\, dx_{j_1} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_s}.$$

$$= (-1)^{2r} f \cdot g\,(x)\, dx_{j_1} \wedge dx_{j_2} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r} \wedge dx_{j_3} \wedge \dots \wedge dx_{j_s}$$

$$\vdots$$

$$= (-1)^{sr} f \cdot g\,(x)\, dx_{j_1} \wedge \dots \wedge dx_{j_s} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_r}$$

$$= (-1)^{sr} \beta^{u}(x) \wedge \alpha^{u}(x)$$

Z

5) Show that $d(\alpha^v)(y) = (\phi \circ \psi^{-1})^* \left(d(\alpha^u)(x) \right)$

Preliminaries

$$\alpha^u(x) = \sum_i a_i(x)\, dx_i$$

$$\alpha^v(y) = \sum_j b_j(y)\, dy_j$$

$$\begin{aligned}
\alpha^v(y) &= (\phi \circ \psi^{-1})^* \left(\alpha^u(x) \right) \\
&= \sum_i a_i(x(y))\, d(x_i(y)) \\
&= \sum_{i,j} a_i(x(y)) \frac{\partial x_i}{\partial y_j}\, dy_j \\
&= \sum_j \underbrace{\left(\sum_i a_i(x(y)) \frac{\partial x_i}{\partial y_j} \right)}_{b_j(y)} dy_j
\end{aligned}$$

LHS

Let's compute $d(\alpha^v)(y)$.

By Definition

$$d(\alpha^{\vee})(y) = \sum_{k,j} \frac{\partial b_j}{\partial y_k} \, dy_k \wedge dy_j$$

Since $b_j = \sum_i a_i(x(y)) \frac{\partial x_i}{\partial y_j}$ we have.

$$\frac{\partial b_j}{\partial y_k} = \sum_i \left(\left(\sum_\ell \frac{\partial a_i}{\partial x_\ell} \frac{\partial x_\ell}{\partial y_k} \frac{\partial x_i}{\partial y_j} \right) + a_i(x(y)) \frac{\partial^2 x_i}{\partial y_k \partial y_j} \right)$$

So

$$d(\alpha^{\vee})(y) = \sum_{k,j} \left(\sum_{i,\ell} \frac{\partial a_i}{\partial x_\ell} \frac{\partial x_\ell}{\partial y_k} \frac{\partial x_i}{\partial y_j} \right) dy_k \wedge dy_j$$

$$+ \sum_{k,j} \left(\sum_i \frac{\partial^2 x_i}{\partial y_k \partial y_j} \right) dy_k \wedge dy_j$$

The second sum vanishes. since

$$\frac{\partial^2 x_i}{\partial y_k \partial y_j} dy_k \wedge dy_j = - \frac{\partial^2 x_i}{\partial y_j \partial y_k} dy_j \wedge dy_k .$$

So $d(\alpha^{v})(y) = \sum_{k,j} \left(\sum_{i,l} \frac{\partial a_i}{\partial x_l} \frac{\partial x_l}{\partial y_k} \frac{\partial x_i}{\partial y_j} \right) dy_k \wedge dy_j$

<u>RHS</u> Now we compute $(\phi \circ \psi^{-1})^*(d(\alpha^u)(x))$.

$$(\phi \circ \psi^{-1})^*(d(\alpha^u)(x)) = (\phi \circ \psi^{-1})^* \left(\sum_{i,j} \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_i \right)$$

$$= \sum_{i,j} \frac{\partial a_i}{\partial x_j}(\cancel{x}(y))\, d(x_j(y)) \wedge d(x_i(y))$$

$$= \sum_{i,j} \frac{\partial a_i}{\partial x_j} \left(\sum_k \frac{\partial x_j}{\partial y_k} dy_k \right) \wedge \left(\sum_l \frac{\partial x_i}{\partial y_l} d\cancel{x}_l \right)$$

$$= \sum_{i,j} \left(\sum_{k,l} \frac{\partial a_i}{\partial x_j} \frac{\partial x_j}{\partial y_k} \frac{\partial x_i}{\partial y_l} dy_k \wedge dy_l \right)$$

$$= \sum_{k,l} \left(\sum_{i,j} \frac{\partial a_i}{\partial x_j} \frac{\partial x_j}{\partial y_k} \frac{\partial x_i}{\partial y_l} \right) d\cancel{y}_k \wedge dy_l$$

$$= \text{LHS}$$