Math 481 Hwk 9 Solutions

1) Assume $\omega = d\sigma$

Then $\int_\alpha \omega = \int_\alpha d\sigma = \int_{\partial\alpha} \sigma = 0$ since $\partial\alpha = 0$

This contradicts $\int_\alpha \omega \neq 0$

2) a) $\partial\gamma_j = \gamma_j(1) - \gamma_j(0)$

$= (1,0,1,0) - (1,0,1,0)$

$= 0$

b) $\int_{\gamma_1} \omega_1 = \int_{[0,1]} \gamma_1^* \omega_1$

$$\gamma_1^* \omega(t) = \frac{-\sin 2\pi t}{1} \, d(\cos 2\pi t) + \frac{\cos 2\pi t}{1} \, d(\sin 2\pi t)$$

$$= 2\pi \sin^2 2\pi t \, dt + 2\pi \cos^2 2\pi t \, dt$$

$$= 2\pi \, dt$$

So $\int_{\gamma_1} \omega_1 = \int_0^1 2\pi \, dt = 2\pi.$

Since $\partial \gamma_1 = 0$ and $\int_{\gamma_1} \omega \neq 0$ it follows from Q1 that $\omega \neq df$ for any function $f: T^2 \to \mathbb{R}$

c) $F^* \omega_1 = \sin^2\theta_1 \, d\theta_1 + \cos^2\theta_1 \, d\theta_1 = d\theta_1$

$d(F^* \omega_1) = d(d\theta_1) = 0$

Assume that $d\omega_1 \neq 0$.
Then for some $p \in T^2$ and $V_1, V_2 \in T_p T^2$ we must have

$d\omega_{1\,(p)}(V_1, V_2) \neq 0$.

Note that $F_* : T_{(\theta_1,\theta_2)}(0,2\pi)^2 \to T_{F(\theta_1,\theta_2)} T^2$.

is 1-1 and onto for all $(\theta_1, \theta_2)$

Hence $V_1 = F_*(W_1)$ and $V_2 = F_*(W_2)$

for some $W_1, W_2 \in T_{F^{-1}(p)} (0, 2\pi)^2$.

Hence we have.

$$\begin{aligned} 0 &\neq d\omega_{1\,(p)} (V_1, V_2) \\ &= d\omega_{1\,(p)} (F_*(W_1), F_*(W_2)) \\ &= (F^* d\omega_1)_{(F^{-1}(p))} (W_1, W_2). \end{aligned}$$

But $F^*(d\omega_1) = d(F^*\omega_1) = 0$

So this is a contradiction.

3) should be $\alpha = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$.

a) $\phi_1 (x, y, z) = (x, y)$

$\phi_1^{-1} (x, y) = (x, y, \sqrt{1 - x^2 - y^2})$

$$(\phi_1^{-1})^* \alpha = x\, dy \wedge d\left(\sqrt{1-x^2-y^2}\right) + y\, d\left(\sqrt{1-x^2-y^2}\right) \wedge dx$$

$$+ \sqrt{1-x^2-y^2}\, dx \wedge dy$$

Now $d\left(\sqrt{1-x^2-y^2}\right) = \dfrac{-x}{\sqrt{1-x^2-y^2}}\, dx + \dfrac{-y}{\sqrt{1-x^2-y^2}}$.

So $$(\phi_1^{-1})^* \alpha = \frac{-x^2}{\sqrt{1-x^2-y^2}}\, dy \wedge dx - \frac{y^2}{\sqrt{1-x^2-y^2}}\, dy \wedge dx$$

$$+ \sqrt{1-x^2-y^2}\, dx \wedge dy$$

$$= \left( \frac{x^2+y^2}{\sqrt{1-x^2-y^2}} + \sqrt{1-x^2-y^2} \right) dx \wedge dy$$

$$= \frac{1}{\sqrt{1-x^2-y^2}}\, dx \wedge dy$$

which does not vanish.