# Math 481 Introduction to Differential Geometry

## Test 2, Tuesday April 28, 2009

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Please read all the questions carefully. You must show all of your work to earn full credit.

1. (15 points) Let $(U, \phi)$ and $(V, \psi)$ be charts on a two-dimensional manifold $M$ such that $U \cap V \neq \emptyset$. Denote the coordinates on the copy of $\mathbb{R}^2$ containing $\phi(U)$ by $(x_1, x_2)$, and the coordinates on $\psi(V)$ by $(y_1, y_2)$.

   (a) Let $W$ be a (0,2)-tensor field on $M$ whose coordinate representative in $\phi(U)$ is
   $$W^U(x_1, x_2) = (x_1 + 5)\, dx_1 \otimes dx_1.$$
   If $y_1 = x_1 + 3x_2$ and $y_2 = -x_1$ compute $W^V(y_1, y_2)$.

$$W^V_{ij} = \sum_{k,l} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j} W^u_{k,l}$$

$$x_1 = -y_1 \qquad x_2 = \frac{1}{3}(y_1 + y_2)$$

$$W^V_{11} = W^V_{12} = W^V_{21} = 0 \qquad W^V_{22} = -y_2 + 5.$$

$$W^V(y_1, y_2) = (-y_2 + 5)\, dy_2 \otimes dy_2.$$

   (b) Define what is means for a (0,2)-tensor field $\omega$ on $M$ to be a 2-form on $M$. Is the tensor field $W$ above a 2-form?

$\omega(V_1, V_2) = -\omega(V_2, V_1)$ for all $V_1, V_2$

$W^u(\partial/\partial x_1, \partial/\partial x_1) = x_1 + 5 \neq 0$ for $x_1 \neq -5$.

So $W$ is not a 2-form

2. (25 points) Consider $\mathbb{R}^3$ with coordinates $(x, y, z)$.

(a) Let $\omega$ be the 1-form on $\mathbb{R}^3$ defined by

$$\omega(x,y,z) = (zx^2 + yx^2)\,dy + (zx^2 + yx^2)\,dz.$$

Compute $d\omega$.

$$d\omega = (z2x + y2x)\,dx \wedge dy + x^2\,\cancel{dz} \wedge dy$$
$$+ (z2x + y2x)\,dx \wedge dz + x^2\,\cancel{dy} \wedge dz$$

$$= 2x(z+y)\,dx \wedge dy + 2x(z+y)\,dx \wedge dz$$

(b) Let $\gamma \colon [0,1]^2 \to \mathbb{R}^3$ be the singular 2-cube defined by

$$\gamma(x_1, x_2) = (x_1 + x_2, x_2, 2).$$

Compute the singular 1-chain $\partial\gamma$.

$$\gamma_{10}(x_1) = \gamma(0, x_1) = (x_1, x_1, 2)$$

$$\gamma_{11}(x_1) = \gamma(1, x_1) = (1 + x_1, x_1, 2)$$

$$\gamma_{20}(x_1) = \gamma(x_1, 0) = (x_1, 0, 2)$$

$$\gamma_{21}(x_1) = \gamma(x_1, 1) = (x_1 + 1, 1, 2)$$

$$\partial\gamma = -\gamma_{10} + \gamma_{11} + \gamma_{20} - \gamma_{21}$$

(c) Compute $\int_\gamma d\omega$.

$$\gamma^* d\omega = 2(x_1+x_2)(2+x_2)\,(\,d(x_1+x_2)\wedge dx_2 + d(x_1+x_2)\wedge 0\,)$$

$$= 2(x_1+x_2)(2+x_2)\,dx_1\wedge dx_2$$

$$\int_\gamma d\omega = \int_{[0,1]^2} \gamma^* d\omega$$

$$= \int_0^1 \int_0^1 2(x_1+x_2)(2+x_2)\,dx_1\,dx_2$$

$$= \int_0^1 2(2+x_2)\left[\frac{x_1^2}{2} + x_2 x_1\right]_0^1 dx_2$$

$$= \int_0^1 2(2+x_2)(\tfrac{1}{2}+x_2)\,dx_2$$

$$= \int_0^1 (2 + 5x_2 + 2x_2^2)\,dx_2$$

$$= \left[2x_2 + \tfrac{5}{2}x_2^2 + \tfrac{2}{3}x_2^3\right]_0^1$$

$$= 2 + \tfrac{5}{2} + \tfrac{2}{3}$$

$$= 31/6$$

(d) Compute $\int_{\partial\gamma} \omega$.

By Stokes' Thm 1.0 we have.

$$\int_{\partial\gamma} \omega = \int_\gamma d\omega = \frac{31}{6}$$

3. (15 points) Let $X$ and $Y$ be vector fields on a manifold $M$. Recall that we have two notations for the Lie derivative of $Y$ with respect to $X$, $\mathcal{L}_X Y$ and $[X, Y]$.

(a) If $p$ is a point in $M$, define the quantity $\mathcal{L}_X Y(p)$.

$$\mathcal{L}_X Y(p) = \lim_{t \to 0} \frac{(\phi_{-t})_* Y(\phi_t(p)) - Y(p)}{t} = \frac{d}{dt}\Big|_{t=0} (\phi_{-t})_* (Y(\phi_t(p)))$$

Here $\phi_t$ is the flow of $X$ near $p$.

(b) Compute $[X, Y]$ for the following two vector fields on $\mathbb{R}^2$:

$$X = (x_1^2 x_2)\frac{\partial}{\partial x_1}$$

and

$$Y = x_1 \sin(x_2)\frac{\partial}{\partial x_1} + x_2 \sin(x_1)\frac{\partial}{\partial x_2}.$$

$$X = \sum v_i \, \partial/\partial x_i \qquad Y = \sum w_j \, \partial/\partial x_j$$

$$[X,Y] = \sum_j \left( \sum_i v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right) \partial/\partial x_j$$

$$= -x_1^2 x_2 (\sin x_1 + \sin x_2)\, \partial/\partial x_1 + x_1^2 x_2^2 \cos x_1 \, \partial/\partial x_2.$$

4. (20 points) Consider the two dimensional sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Let $\omega$ be the 2-form on $S^2$ defined by restricting to $S^2$ the following form on $\mathbb{R}^3$

$$x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

The form $\omega$ is nonvanishing and hence determines an orientation on $S^2$.

(a) Consider the singular 2-cube $\gamma\colon [0,1]^2 \to S^2$ defined by

$$\gamma(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_2}{2}, \sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}\right)$$

Determine whether or not $\gamma$ is orientation preserving.

Note $\gamma^* dx = \frac{1}{2} dx_1 \quad \gamma^* dy = \frac{1}{2} dx_2$

$$\gamma^* dz = \frac{\frac{-x_1}{4}}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} dx_1 + \frac{\frac{-x_2}{4}}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} dx_2.$$

So $$\gamma^* \omega = -\frac{x_1^2}{16} \frac{1}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} dx_2 \wedge dx_1$$

$$- \frac{x_2^2}{16} \frac{1}{\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} dx_2 \wedge dx_1$$

$$+ \frac{1}{4} \sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}\, dx_1 \wedge dx_2.$$

$$= \frac{1}{4\sqrt{1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}}} dx_1 \wedge dx_2$$

since this function ↑ is positive $\gamma$ is orientation preserving.

(b) Compute $\int_V \omega$ where $V$ is the image of the singular 2-cube $\gamma$ from part (a). You can leave your answer in the form of a definite double integral.

By (a) we have.

$$\int_V \omega = \int_\gamma \omega = \int_{[0,1]^2} \gamma^* \omega$$

$$= \int_0^1 \int_0^1 \frac{1}{4\sqrt{1-\frac{x_1^2}{4}-\frac{x_2^2}{4}}} \, dx_1 \, dx_2 .$$

5. (15 points) Consider $\omega \in \Lambda^r(M)$ and $\eta \in \Lambda^s(M)$ for $r, s \geq 1$.

(a) Prove that if $d\omega = 0$ and $d\eta = 0$ then $d(\omega \wedge \eta) = 0$.

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$$
$$= 0 \wedge \eta + (-1)^r \omega \wedge 0$$
$$= 0$$

(b) Prove that if $\omega = d\alpha$ and $d\eta = 0$ then $\omega \wedge \eta = d\xi$ for some $\xi \in \Lambda^{r+s-1}(M)$.

choose $\xi = \alpha \wedge \eta$.

$$d\xi = d(\alpha \wedge \eta) = d\alpha \wedge \eta + (-1)^{r-1} \alpha \wedge d\eta$$
$$= \omega \wedge \eta + 0$$
$$= \omega \wedge \eta$$

(c) Find a 2-form $\omega$ on $\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

$$\omega \wedge \omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + 0 + 0 + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + (-1)^4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$= 2\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$\neq 0.$$