Homework #6

1) 5.1 #2

$M: \quad z = f(x,y)$

We have the (global) coord. patch $\bar{x}(u,v) = (u, v, f(u,v))$

a) $\bar{x}(0,0) = (0, 0, f(0,0)) = (0,0,0)$

$\bar{x}_u(0,0) = (1, 0, f_x(0,0)) = (1,0,0) = U_1(\bar{0})$

$\bar{x}_v(0,0) = (0, 1, f_y(0,0)) = (0,1,0) = U_2(\bar{0})$

So $U_1(\bar{0})$ and $U_2(\bar{0})$ belong to $T_{\bar{0}}M$.

Also $U = \dfrac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1 + f_x^2 + f_y^2}}$

b) $S(\bar{u}_1) = -\nabla_{\bar{u}_1} U$

$= -\nabla_{\bar{x}_u} U$

$$S(\bar{u}_1) = -\left\langle \frac{\partial}{\partial u}\left(\frac{-f_x}{\sqrt{1+f_x^2+f_y^2}}\right), \frac{\partial}{\partial u}\left(\frac{-f_y}{\sqrt{1+f_x^2+f_y^2}}\right), \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{1+f_x^2+f_y^2}}\right)\right\rangle$$

$$= \langle f_{xx}(\bar{0}), f_{xy}(\bar{0}), 0\rangle.$$

Similarly $\quad S(\bar{u}_2) = \langle f_{yx}(\bar{0}), f_{yy}(\bar{0}), 0\rangle.$

3) 5.1 #4

a) $x^2+y^2=r^2$

$$u = \frac{\vec{\nabla} F}{\|\vec{\nabla} F\|} = \frac{\langle 2x, 2y, 0\rangle}{\sqrt{4x^2+4y^2}} = \langle \tfrac{x}{r}, \tfrac{y}{r}, 0\rangle$$

$G(M)$ is the equator.

d) $(x-1)^2+y^2+(z+2)^2=1$

$$u = \frac{\vec{\nabla} F}{\|\vec{\nabla} F\|} = \frac{2\langle x-1, y, z+2\rangle}{2\sqrt{(x-1)^2+y^2+(z+2)^2}} = \langle x-1, y, z+2\rangle.$$

$G(M)$ is the whole sphere.

5) 5.2 #3

$\alpha_n = (r\cos t, r\sin t, \pm t^n)$

$\alpha_n' = (-r\sin t, r\cos t, \pm n t^{n-1})$

$\alpha_n'' = (-r\cos t, -r\sin t, \pm n(n-1)t^{n-2})$

$$\alpha_n''(0) = \begin{cases} (-r, 0, 0) & ; n > 2 \\ (-r, 0, 2) & ; n = 2. \end{cases}$$

$U = \frac{1}{r}\langle x, y, 0\rangle.$

$U(\alpha_n(0)) = U(r, 0, 0) = \frac{1}{r}\langle r, 0, 0\rangle$

$\alpha_n''(0) \cdot U = -r$ for all $n \geq 2$.

7) 5.3 #5

$\hat{C}_0 : \quad k_1 x^2 + k_2 y^2 = 0$

a) i) $K(p) > 0 \Rightarrow k_1, k_2 > 0$ or $k_1, k_2 < 0$.

In either case $\hat{C}_0$ is the point $(0,0)$

ii) $K(p) < 0 \Rightarrow k_1 > 0$ and $k_2 < 0$

$$\Rightarrow x = \pm\sqrt{\frac{-k_2}{k_1}}\, y$$

$\hat{C}_0$ is a pair of lines through the origin

iii) $K(p) = 0 \Rightarrow k_1 = 0$ and/or $k_2 = 0$

$\hat{C}_0$ is one ~~or both~~ of the coordinate axes.

or the whole plane.