Math 423 Differential Geometry

Assignment 7, Due Tuesday Dec 8

Throughout these exercises Σ is a surface, and for $p \in \Sigma$ and some some $\epsilon > 0$ the map $\exp_p \colon B_\epsilon \subset T_p\Sigma \to \Sigma$ is a regular surface element.

1. In class we defined geodesic polar coordinates. In this question we will consider geodesic normal coordinates.

   Choose two orthogonal unit vectors e_1 and e_2 in $T_p\Sigma$. Every element $v \in B_\epsilon$ can be expressed uniquely in the form $v = u_1 e_1 + u_2 e_2$. We say that u_1 and u_2 are geodesic normal coordinates centered at p.

   (a) Show that in geodesic normal coordinates centered at p we have $E(p) = 1 = G(p)$ and $F(p) = 0$.

   (b) Show that in geodesic normal coordinates centered at p all of the Christoffel symbols (of the second kind) vanish at p.

2. Show that in geodesic polar coordinates the Gaussian curvature is

$$K(\rho, \theta) = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}.$$

3. Recall that a geodesic circle on Σ is the image of a circle

$$\{v \in T_p\Sigma \mid \|v\| = \delta\}$$

   in B_ϵ under the map $\exp_p$. Prove that on a surface of constant curvature the geodesic circles have constant geodesic curvature.

4. Show that in geodesic polar coordinates the geodesic equations are given by

$$\begin{aligned} \rho'' - \frac{1}{2}(\theta')^2 G_\rho &= 0 \\ \theta'' + \frac{G_\rho}{G}\rho'\theta' + \frac{1}{2}(\theta')^2 \frac{G_\theta}{G} &= 0. \end{aligned}$$