Math 423 Solutions to Exam 2

1) $f(u^1, u^2) = (u^1 \cos(u^2), u^1 \sin(u^2), u^2)$

a) $f_1 = (\cos u^2, \sin u^2, 0)$

$f_2 = (-u^1 \sin u^2, u^1 \cos u^2, 1)$

$f_1 \times f_2 = (\sin u^2, -\cos u^2, u^1)$

$\nu = (\sin u^2, -\cos u^2, u^1) \dfrac{1}{\sqrt{1+(u^1)^2}}$

$f_{11} = (0, 0, 0)$

$f_{12} = (-\sin u^2, \cos u^2, 0)$

$f_{22} = (-u^1 \cos u^2, -u^1 \sin u^2, 0)$

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & (u^1)^2+1 \end{pmatrix}$$

$$(h_{ij}) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+(u')^2}} \\ -\frac{1}{\sqrt{1+(u')^2}} & 0 \end{pmatrix}$$

b) $$K = \frac{\det(h_{ij})}{\det(g_{ij})} = -\frac{1}{(1+(u')^2)^2}$$

$$H = \frac{1}{2\det(g_{ij})}\left(\overset{0}{h_{11}}\, g_{22} + \overset{0}{h_{22}}\, g_{11} - 2\,\overset{0}{h_{12}}\, g_{12}\right)$$

$$= 0$$

$$H = \frac{1}{2}(K_1 + K_2) \Rightarrow K_1 = -K_2$$

$$K = K_1 \cdot K_2$$

$$\Rightarrow -K_1^2 = -\frac{1}{(1+(u')^2)^2}$$

$$\Rightarrow K_1 = -\frac{1}{1+(u')^2} \qquad K_2 = \frac{1}{1+(u')^2}.$$

c) $$\Gamma^k_{ij} = \frac{1}{2} \sum_\ell \left(\frac{\partial g_{j\ell}}{\partial u^i} + \frac{\partial g_{i\ell}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\ell} \right) g^{k\ell}.$$

<u>$k=1$</u> $$\Gamma^1_{ij} = \frac{1}{2} \left(\frac{\partial g_{j1}}{\partial u^i} + \frac{\partial g_{i1}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^1} \right) g^{11}$$

$\Rightarrow \Gamma^1_{ij} = 0$ unless $i=j=2$ in which case.

$$\Gamma^1_{22} = \frac{1}{2}\left(0 + 0 - 2u^1 \right) \cdot 1 = -u^1$$

<u>$k=2$</u> $$\Gamma^2_{ij} = \frac{1}{2} \left(\frac{\partial g_{j2}}{\partial u^i} + \frac{\partial g_{i2}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^2} \right) g^{22}.$$

$\Rightarrow \Gamma^2_{ij} = 0$ ~~unless~~ unless $i=1$ & $j=2$ or $i=2$ & $j=1$.

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}(2u^1) \frac{1}{1+(u^1)^2} = \frac{u_1}{1+(u^1)^2}.$$

$$\begin{cases} (\ddot{u}^1) + (\dot{u}^2)^2(-u^1) = 0 & \text{①} \\ (\ddot{u}^2) + 2(\dot{u}^1)(\dot{u}^2)\dfrac{u^1}{1+(u^1)^2} = 0 & \text{②} \end{cases}$$

d) If $\mathring{u}^1 = 0$ we have.

$$\begin{cases} (\mathring{u}^2)^2 u^1 = 0 \\ (\ddot{u}^2) = 0 \end{cases} \Rightarrow \begin{cases} a = 0, u^1 = c \quad \underline{\text{or}} \quad u^1 = 0 \\ u^2(t) = at + b. \end{cases}$$

$$\Rightarrow \begin{cases} u^1 = c \quad u^2 = b \\ u^1 = 0 \quad u^2 = at + b \end{cases}$$

The first alternative yields a constant curve. ie a trivial geodesic.

The second alternative yields.

$c(t) = (0, 0, at + b)$ a vertical line.

If $\mathring{u}^2 = 0$ we have.

$$\begin{cases} (\ddot{u}_1) = 0 \\ 0 = 0 \end{cases} \Rightarrow u_1(t) = (at + b) \quad u_2 = c.$$

This yields geodesics of the form.

$$c(t) = \big((at+b)\cos c \,,\ (at+b)\sin c \,,\ c \big)$$

which are horizontal straight lines.

2) <u>Thm Egregium</u> The Gauss curvature of a surface element is intrinsic, i.e. K. depends only on the g_{ij}'s and their partial derivatives

Computing $\tilde{g}_{ij}$ for $\tilde{f}$ we get:

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1+(u^1)^2 \end{pmatrix} = g_{ij}$$

Hence $\tilde{K} = K$ by Thm Egregium.