Math 518 Differential Manifolds I

Assignment 1, Due Tuesday Sept 8

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1. Consider two copies of the real line with coordinates s and t respectively. The quotient space corresponding to the equivalence relation

$$s \sim t \iff s = t \neq 0$$

is called *the real line with two origins.* Show that it is locally homeomorphic to $\mathbb{R}$ but is not Hausdorff. Define, in the same manner, a real line with uncountably many origins, and show that it is also not second countable.

2. Prove that the union of the two coordinate axes in $\mathbb{R}^2$ is not a manifold. (Hint: Consider what happens to a neighborhood of the origin when the origin is removed.)

3. Let $X \subset \mathbb{R}^N$ be a smooth k-dimensional manifold in the sense of Guillemin and Pollack. Show that X is also a manifold according to the definition given in class.

4. Construct a smooth atlas on the unit square

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid \max\{|x_1|, |x_2|\} = 1\}.$$

5. (Bump functions)

   (a) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

   Prove that f is smooth.

   (b) For $a < b$, prove that $g(x) = f(x-a)f(b-x)$ is smooth, positive on (a, b) and zero elsewhere. Draw the graph of the function

$$h(x) = \frac{\int_{-\infty}^{x} g(x)\,dx}{\int_{-\infty}^{\infty} g(x)\,dx}.$$

   (c) Construct a smooth function $\mathbb{R}^k$ which equals 1 on the ball of radius a, equals zero outside the ball of radius b, and is strictly between 0 and 1 elsewhere.