Math 518 Differential Manifolds I

Assignment 4, Due Thursday October 7

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1. For a submanifold L of M let $i\colon L \to M$ be the inclusion map. Prove that $i_*(x)$ is the inclusion map of T_xL into T_xM.

2. If U is an open subset of M prove that $T_xU = T_xM$ for all $x \in U$.

3. Suppose that $F\colon M \to N$ is a diffeomorphism, Prove that $F_*(x)$ is an isomorphism of tangent spaces for all x in M.

4. (a) For any two manifolds M and N show that for all $(x, y) \in M \times N$

$$T_{(x,y)}M \times N = T_xM \times T_yN.$$

   (b) Let $\pi\colon M \times N \to M$ be projection. Prove that

$$\pi_*(x)\colon T_xM \times T_yN \to T_xM$$

   is the analogous projection map.

   (c) Let $F\colon M \to M'$ and $G\colon N \to N'$ be smooth maps. Prove that

$$(F \times G)_*(x, y) = F_*(x) \times G_*(y).$$

5. Let M be a smooth manifold. A section of the tangent bundle $\pi\colon TM \to M$ (also called a vector field) is a smooth map $s\colon M \to TM$ with the property that $\pi \circ s$ is the identity on M.

   (a) Prove that a section $s\colon M \to TM$ is an embedding.

   (b) We can define $\sigma(x) = 0_x$, the zero vector in T_xM for all $x \in M$. This is called the zero section. Prove that for every $x \in M$, there exists a section $s\colon M \to TM$ so that $s(x) \neq 0_x$.

6. Let $g\colon M \to \mathbb{R}$ be a smooth function which is positive everywhere. Show that the map $TM \to TM$ defined by $(x, v) \mapsto (x, g(x)v)$ is smooth.

7. Consider two smooth maps $F\colon M \to N$ and $G\colon N \to P$. Let Y be a submanifold of P. If G is transveral to Y show that $F \pitchfork G^{-1}(Y)$ iff $G \circ F \pitchfork Y$.

   (Aside: the Latex command for the tranversal symbol is "pitchfork")

8. For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What do the intersections look like for different values of a?

9. Let V be a vector space and let Δ be the diagonal in $V \times V$. For a linear map $A\colon V \to V$ consider the graph

$$W = \{(v, Av) \mid v \in V\}.$$

Show that $W \pitchfork \Delta$ iff 1 is not an eigenvalue of A.

10. Let V and W be subspaces of $\mathbb{R}^n$, and let

$$q\colon \mathbb{R}^n \smallsetminus \{0\} \to \mathbb{R}P^{n-1}$$

be the usual quotient map.

(a) Prove that for any subspace U of $\mathbb{R}^n$, $q(U \smallsetminus \{0\})$ is a submanifold of $\mathbb{R}P^{n-1}$ of dimension equal to $\mathrm{dim(U)} - 1$

(b) Show that $V \pitchfork W$ iff $\mathrm{dim(V)} + \mathrm{dim(W)} - \mathrm{dim(V \cap W)} = \mathrm{n}$.

(c) Show that if $V \pitchfork W$ then $q(V \smallsetminus \{0\}) \pitchfork q(W \smallsetminus \{0\})$.