HW 6 SOLUTIONS, MA518

1. Problem 1

Since $dim(V) = m$, we have $dim(\Lambda^m(V)) = 1$ with generator $\omega_0 = dx_1 \wedge \cdots \wedge dx_m$ in the coordinates on V with respect to a basis $(e_1, \cdots, e_m)$. So it is enough to compute $A^*\omega_0$. Let $A = (a_{ij})$ be the matrix form. We have $A^*dx_i(e_k) = dx_i(Ae_k) = a_{ik}$. So

$$A^*dx_i = \sum_k a_{ik}dx_k$$

If S_m denotes the group of permutations over m letters, then

$$\begin{aligned}
A^*\omega_0 &= \left(\sum a_{1k}dx_k\right) \wedge \cdots \wedge \left(\sum_k a_{mk}dx_k\right) \\
&= \sum_{\sigma \in S_m} (a_{1\sigma(1)} \cdots a_{m\sigma(m)})dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(m)} \\
&= \sum_{\sigma \in S_m} (-1)^{sgn(\sigma)}(a_{1\sigma(1)} \cdots a_{m\sigma(m)})dx_1 \wedge \cdots \wedge dx_m \\
&= \det(A)\omega_0
\end{aligned}$$

So $A^*\omega = \det(A)\omega$.

Remark 1.1. Suppose f is a diffeomorphism from an open set U in $\mathbb{R}^m$ to some other open set V. Then by a similar sort of calculation, you can check that when the standard volume form $\omega = dx_1 \wedge \cdots \wedge dx_m$ is pulled back by f, it gets multiplied by the Jacobian of f, i.e. if $y = f(x)$

$$df^*\omega(x) = \det(Df(y))\omega$$

This illustrates the main point about differential forms: the change of variables formula gets automatically built in because of the above equation, and so the integral of a differential form is well-defined and independent of the choice of co-ordinates. This allows us to integrate on manifolds with no reference to any embedding in some Euclidean space.

2. Problem 2

When V is 1-dimensional, $\Lambda^2(V) = 0$.

When V is 2-dimensional, $\Lambda^2(V)$ is one dimensional spanned by $dx_1 \wedge dx_2$. So it is obviously decomposable, and hence so is $f(\mathbf{x})dx_1 \wedge dx_2$.

When V is 3-dimensional, $\Lambda^2(V)$ is spanned by the forms $dx_1 \wedge dx_2, dx_2 \wedge dx_3, dx_3 \wedge dx_1$. Suppose at a point $\mathbf{x} \in V$, we have a 2-form $\omega = a_{12}dx_1 \wedge dx_2 + a_{23}dx_2 \wedge dx_3 + a_{31}dx_3 \wedge dx_1$. If it were to be decomposable, we should be able to write it as

$$\begin{aligned}
\omega &= (b_1dx_1 + b_2dx_2 + b_3dx_3) \wedge (c_1dx_1 + c_2dx_2 + c_3dx_3) \\
&= (b_1c_2 - b_2c_1)dx_1 \wedge dx_2 + (b_2c_3 - b_3c_2)dx_2 \wedge dx_3 + (b_3c_1 - b_1c_3)dx_3 \wedge dx_1
\end{aligned}$$

This gives us equations

$$\begin{aligned} b_1c_2 - b_2c_1 &= a_{12} \\ b_2c_3 - b_3c_2 &= a_{23} \\ b_3c_1 - b_1c_3 &= a_{31} \end{aligned}$$

Check directly that the above system always has solutions.

Consider now the same problem in $\mathbb{R}^4$, and let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. We claim that this is not decomposable. Suppose on the contrary that it is decomposable. Write it as

$$\omega = \left(\sum b_i dx_i\right) \wedge \left(\sum c_i dx_i\right) = \alpha_1 \wedge \alpha_2$$

First, note that all b_i and c_i have to be nonzero. For instance, if $b_1 = 0$ then to have the term $dx_1 \wedge dx_2$ in ω we must have $c_1 \neq 0$. But then ω also contains terms linear in $dx_1 \wedge dx_3$ and $dx_1 \wedge dx_4$.

Denote the inclusion of $\mathbb{R}^2$ in $\mathbb{R}^4$ as a subspace spanned by the basis elements $\{e_i, e_j\}$ by V_{ij}. The pullback of ω to V_{13} by its inclusion map is 0. The pullback of the 1-forms α_1 and α_2 to V_{13} are $b_1dx_1 + b_3dx_3$ and $c_1dx_1 + c_3dx_3$. Since the wedge product commutes with pullbacks, the forms $b_1dx_1+b_3dx_3$ and $c_1dx_1+c_3dx_3$ are linearly dependent. Applying similar logic to other such subspaces we get that the corresponding pullback one forms are linearly dependent. Since all b_i and c_i are non-zero, this forces $\alpha_2 = r\alpha_1$. But then, $\alpha_1 \wedge \alpha_2 = 0$, a contradiction.

3. Problem 6

Suppose ω is a 1-form on S_1 and let

$$I_\omega = \int_{S^1} \omega$$

Let

$$I_\nu = \int_{S^1} \nu$$

and set $c = I_\omega / I_\nu$ which is well-defined since $I_\nu \neq 0$. Check that

$$\int_{S^1} (\omega - c\nu) = 0$$

By Problem 5, there is a function f such that $\omega - c\nu = df$.