HW 7 SOLUTIONS, MA518

Problem 1

(a): The Lie derivative of ω is the differential form given by

$$\mathcal{L}_X\omega = \lim_{t\to 0}\frac{\phi_t^*\omega - \omega}{t}$$

For vector fields $Y_1, \cdots, Y_k$ this means

$$\mathcal{L}_X\omega(Y_1,\cdots,Y_k) = \lim_{t\to 0}\frac{\omega(D\phi_t(Y_1),\cdots,D\phi_t(Y_k)) - \omega(Y_1,\cdots,Y_k)}{t}$$

Now we have the following computation: First,

$$\begin{aligned}\omega(D\phi_t(Y_1),\cdots,D\phi_t(Y_k)) - \omega(Y_1,\cdots,Y_k) &= \omega(D\phi_t(Y_1),\cdots,D\phi_t(Y_k)) - \omega(Y_1\circ\phi_t,\cdots,Y_k\circ\phi_t)\\ &\quad + \omega(Y_1\circ\phi_t,\cdots,Y_k\circ\phi_t) - \omega(Y_1,\cdots,Y_k)\end{aligned}$$

Note, from the definition of flow

$$\lim_{t\to 0}\frac{\omega(Y_1\circ\phi_t,\cdots,Y_k\circ\phi_t) - \omega(Y_1,\cdots,Y_k)}{t} = X\omega(Y_1,\cdots,Y_k)$$

So, we are reduced to computing the quantity

$$S = \lim_{t\to 0}\frac{\omega(D\phi_t(Y_1),\cdots,D\phi_t(Y_k)) - \omega(Y_1\circ\phi_t,\cdots,Y_k\circ\phi_t)}{t}$$

Notice that by the definition of pull-back of vector fields under diffeomorphisms $D\phi_t(Y_i) = \phi_{-t}^*Y_i$. So

$$S = \lim_{t\to 0}\frac{\omega(\phi_{-t}^*Y_1,\cdots,\phi_{-t}^*Y_k) - \omega(Y_1\circ\phi_t,\cdots,Y_k\circ\phi_t)}{t}$$

Now using a telescoping sum

$$\begin{aligned}S = \lim_{t\to 0}\frac{1}{t}[&\omega(\phi_{-t}^*Y_1,\cdots,\phi_{-t}^*Y_k) - \omega(Y_1\circ\phi_t,\phi_{-t}^*Y_2,\cdots,\phi_{-t}^*Y_k)\\ &+ \omega(Y_1\circ\phi_t,\phi_{-t}^*Y_2,\cdots,\phi_{-t}^*Y_k) - \omega(Y_1\circ\phi_t,Y_2\circ\phi_t,\phi_{-t}^*Y_3,\cdots,\phi_{-t}^*Y_k)\\ &+\cdots\\ &+\cdots\\ &+ \omega(Y_1\circ\phi_t,\cdots,Y_{k-1}\circ\phi_t,\phi_{-t}^*Y_k) - \omega(Y_1\circ\phi_t,\cdots,Y_k\circ\phi_t)]\end{aligned}$$

In other words, we have

$$S = \lim_{t\to 0}\frac{1}{t}\left(\sum_{j=1}^{k}\omega(Y_1\circ\phi_t,\cdots,Y_{j-1}\circ\phi_t,\phi_{-t}^*Y_j,\cdots,\phi_{-t}^*Y_k) - \omega(Y_1\circ\phi_t,\cdots,Y_j\circ\phi_t,\phi_{-t}^*Y_{j+1},\cdots,\phi_{-t}^*Y_k)\right)$$

By definition of Lie derivative for vector fields, each term

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\left(\omega(Y_1\circ\phi_t,\cdots,Y_{j-1}\circ\phi_t,\phi_{-t}^*Y_j,\cdots,\phi_{-t}^*Y_k) - \omega(Y_1\circ\phi_t,\cdots,Y_j\circ\phi_t,\phi_{-t}^*Y_{j+1},\cdots,\phi_{-t}^*Y_k)\right)\\ &\quad= \omega(Y_1,\cdots,Y_{j-1},[-X,Y_j],Y_{j+1},\cdots,Y_k)\\ &\quad= (-1)^j\omega([X,Y_j],Y_1,\cdots,\hat{Y}_j,\cdots,Y_k)\end{aligned}$$

Putting the computation together

$$\mathcal{L}_X\omega = X\omega(Y_1,\cdots,Y_k) + \sum(-1)^j\omega([X,Y_j],Y_1,\cdots,\hat{Y_j},\cdots,Y_k)$$

(b): The exterior derivative is given by

$$\begin{aligned} d\omega(Y_0,\cdots,Y_k) &= \sum(-1)^i Y_i\omega(Y_0,\cdots,\hat{Y_i},\cdots,Y_k) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([Y_i,Y_j],Y_0,\cdots,\hat{Y_i},\cdots,\hat{Y_j},\cdots,Y_k) \end{aligned}$$

So

$$\begin{aligned} d\imath_X\omega(Y_1,\cdots,Y_k) &= \sum(-1)^{i-1} Y_i\omega(X,Y_1,\cdots,\hat{Y_i},\cdots,Y_k) \\ &+ \sum_{i<j}(-1)^{i+j}\omega(X,[Y_i,Y_j],\cdots,\hat{Y_i},\cdots,\hat{Y_j},\cdots,Y_k) \end{aligned}$$

Similarly

$$\begin{aligned} \imath_X d\omega(Y_1,\cdots,Y_k) &= X\omega(Y_1,\cdots,Y_k) \\ &+ \sum(-1)^i Y_i\omega(X,Y_1,\cdots,\hat{Y_i},\cdots,Y_k) \\ &+ \sum(-1)^j\omega([X,Y_j],Y_1,\cdots,\hat{Y_j},\cdots,Y_k) \\ &+ \sum(-1)^{i+j}\omega([Y_i,Y_j],X,Y_1,\cdots,\hat{Y_i},\cdots,\hat{Y_j},\cdots,Y_k) \end{aligned}$$

Finally, notice that the two sums in the expression for $d\imath_X\omega$ cancel with the first and third sums in $\imath_X d\omega$. Hence

$$(d\imath_X + \imath_X d)\omega(Y_1,\cdots,Y_k) = X\omega(Y_1,\cdots,Y_k) + \sum(-1)^j\omega([X,Y_j],Y_1,\cdots,\hat{Y_j},\cdots,Y_k)$$

Comparing parts (a) and (b), we have shown $\mathcal{L}_X\omega = (d\imath_X + \imath_X d)\omega$.

Problem 2

Consider the 1-form $\alpha = f dx + g dy$. By the generalized Stokes

$$\int_\gamma \alpha = \int_W d\alpha$$

Computing

$$d\alpha = \left(\frac{\partial f}{\partial y}\right) dy\wedge dx + \left(\frac{\partial g}{\partial x}\right) dx\wedge dy = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\wedge dy$$

which indicates how the classical Green's formula is a special case of the Stokes.

Problem 3

Since N has dimension $m-1$, we have $d\omega = 0$. Since the exterior derivative commutes with pull-back $d(F^*\omega) = F^*(d\omega) = 0$ i.e $\alpha = F^*\omega$ is a closed form. By the generalized Stokes theorem

$$\int_{\partial M}\alpha = \int_M d\alpha = 0$$

Problem 4

For $t \in I = [0,1]$, let F_t be the homotopy between F_0 and F_1. In other words, there is a map $G : M \times I \to N$ such that $G(t,x) = F_t(x)$. Applying Problem 3 to the map G,

$$\int_{\partial(M\times I)} G^*\omega = 0$$

The boundary $\partial(M \times I) = M \times \partial I$ which is two disjoint copies of M with opposite orientations.

$$\int_{\partial(M\times I)} G^*\omega = \int_M (F_1^*\omega - F_0^*\omega) = 0$$

proving whats required.

Problem 5

Consider S^2 as ∂B^3. By generalized Stokes Theorem

$$\int_{S^2} \alpha = \int_{B^3} d\alpha$$

Check that $d\alpha = dx \wedge dy \wedge dz$. So

$$\int_{S^2} \alpha = \int_{B^3} dx \wedge dy \wedge dz = vol(B^3) = \frac{4\pi}{3}$$