HW 1 SOLUTIONS, MA518

1. Problem 2

Suppose that the union of the two co-ordinate axes is a manifold. Let $f : (-1,1) \to U$ be a co-ordinate chart around the point $p = (0,0)$ in $\mathbb{R}^2$. Consider the restriction $f : (-1,1) \setminus \{0\} \to U \setminus p$. It is continuous, so the number of connected components of the image of the restriction has to be less than or equal to the number of connected components of the domain. A contradiction.

2. Problem 5

(a): The only issue is smoothness at $x = 0$. This can be checked from first principles. It reduces to checking

$$\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x^n} = 0$$

for all positive integers n.

(b): It is straightforward to check from the definition of g that it is smooth (because f is smooth), positive on (a,b) and zero elsewhere. The function $h(x)$ is smooth on $\mathbb{R}$, zero for $x \leq a$, 1 for $x \geq b$ and monotonically increasing on (a,b).

(c): Let $\mathbf{x}$ denote a point in $\mathbb{R}^k$ and let $r = \|\mathbf{x}\|$ be the Euclidean norm. Define $\phi(\mathbf{x}) = 1 - h(r)$. Check that ϕ has the required properties.