HW 2 SOLUTIONS, MA518

1. Problem 1

Let p be a point in U. Since F is a submersion, by the implicit function theorem, there are local co-ordinates $(x_1, \cdots, x_n)$ around p (i.e. with $p = (0, \cdots, 0)$) and local co-ordinates $(x_1, \cdots, x_m)$ around $F(p)$ (i.e. with $F(p) = (0, \cdots, 0)$) such that $F(x_1, \cdots, x_n) = (x_1, \cdots, x_m)$ in the local co-ordinates i.e. in the local co-ordinates, F is just the projection onto the first m co-ordinates. By shrinking the co-ordinate chart if necessary we can assume that it lies entirely inside U.

A projection onto the first m co-ordinates is an open map. So the open set given by the co-ordinate chart maps under F, to an open set in N. Since this can be done at every point p in U, the image $F(U)$ is open.

2. Problem 3

There are global co-ordinates on $M(n)$ by identifying the (i, j)-th entry of a matrix x_{ij} with the corresponding co-ordinate in $\mathbb{R}^{n^2}$. Let S_n be the group of permutations over n letters. The determinant in the above co-ordinates has the expression

$$\det(M) = \sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} x_{1\sigma(1)} \cdots x_{2\sigma(2)}$$

To compute the derivative of the determinant

$$\begin{aligned} (D\det)_M \left(\frac{\partial}{\partial x_{ij}} \right) &= \left(\frac{\partial}{\partial x_{ij}} \det(M) \right) \frac{\partial}{\partial t} \\ &= \left(\sum_{\sigma(i)=j} (-1)^{sgn(\sigma)} x_{1\sigma(1)} \cdots \widehat{x_{ij}} \cdots x_{n\sigma(n)} \right) \frac{\partial}{\partial t} \end{aligned}$$

where the notation $\widehat{x_{ij}}$ means that the term x_{ij} is omitted from the product. It is easy to check that

$$\sum_{\sigma(i)=j} (-1)^{sgn(\sigma)} x_{1\sigma(1)} \cdots \widehat{x_{ij}} \cdots x_{n\sigma(n)} = C_{ij}$$

where C_{ij} is the (i, j)-th cofactor of M. Actually, the above calculation for the derivative evaluated on $\partial/\partial x_{ij}$ can be directly seen using the co-factor expansion for the i-th row. So we have

$$\begin{aligned}(D\det)_M(M) &= \sum_{ij} x_{ij}(D\det)_M\left(\frac{\partial}{\partial x_{ij}}\right) \\ &= \sum_{ij} x_{ij}C_{ij}\frac{\partial}{\partial t} \\ &= \det(M)\frac{\partial}{\partial t} \neq 0\end{aligned}$$

This shows that any non-zero number in $\mathbb{R}$ is a regular value for the det function on $M(n)$.

3. Problem 5

Consider the map $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by $f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 + x_3^2 + x_4^2, x_1x_2x_3)$. Denote the co-ordinates on $\mathbb{R}^2$ by (y_1, y_2). Compute the derivative

$$\begin{aligned}df(\frac{\partial}{\partial x_1}) &= \frac{\partial}{\partial x_1}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_1}(x_1x_2x_3)\frac{\partial}{\partial y_2} \\ &= 2x_1\frac{\partial}{\partial y_1} + x_2x_3\frac{\partial}{\partial y_2}\end{aligned}$$

A similar calculation gives

$$\begin{aligned}df(\frac{\partial}{\partial x_2}) &= 2x_2\frac{\partial}{\partial y_1} + x_1x_3\frac{\partial}{\partial y_2} \\ df(\frac{\partial}{\partial x_3}) &= 2x_3\frac{\partial}{\partial y_1} + x_1x_2\frac{\partial}{\partial y_2} \\ df(\frac{\partial}{\partial x_4}) &= 2x_4\frac{\partial}{\partial y_1}\end{aligned}$$

We figure out the constraints that get imposed if we insist that the vectors $df(\partial/\partial x_i)$ be linearly dependent, while requiring that the point be in the pre-image of $(y_1, y_2) = (1, 1)$. The points in the pre-image satisfy x_1, x_2 and $x_3 \neq 0$ because otherwise $y_2 = 0$. Linear dependence implies that

$$\frac{x_2x_3}{x_1} = \frac{x_3x_1}{x_2} = \frac{x_1x_2}{x_3}$$

which is equivalent to $x_1 = x_2 = x_3$. This implies x_1, x_2 and x_3 are all less than or equal to $1/3$ (from the requirement that $y_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$), hence the product $x_1x_2x_3$ cannot be 1. Thus the vectors cannot be all linearly dependent.

This means that $(y_1, y_2) = (1, 1)$ is a regular value for f. Hence the pre-image is a smooth sub-manifold of $\mathbb{R}^4$.