

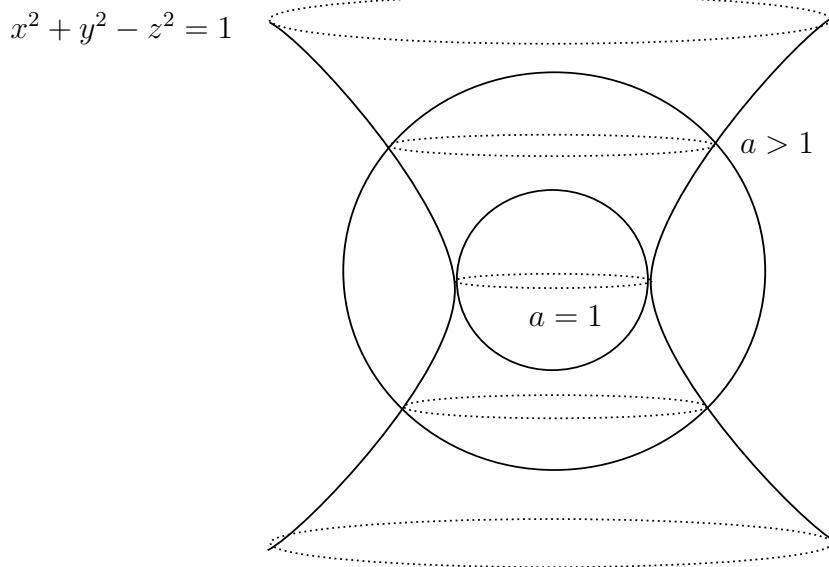
HW 4 SOLUTIONS, MA518

1. PROBLEM 8

Solving for intersection, we get $2z^2 = a - 1$. This has real solutions only for $a \geq 1$. We separate into 2 cases: $a = 1$ and $a > 1$. When $a > 1$ the intersection is

$$x^2 + y^2 = (a + 1)/2, z = \pm \sqrt{\frac{a - 1}{2}}$$

which are two circles in the two horizontal planes given by the second equation above. When $a = 1$, the intersection is a single circle $x^2 + y^2 = 1$ in the $x - y$ plane.



Let $f(x, y, z) = x^2 + y^2 - z^2$. The tangent space at (x, y, z) to $x^2 + y^2 - z^2 = 1$ is the kernel of the linear map df . This is the subspace spanned by

$$u_1 = \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \quad , \quad u_2 = \left(2xy \frac{\partial}{\partial z} + yz \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} \right)$$

Let $g(x, y, z) = x^2 + y^2 + z^2$. The tangent space at (x, y, z) to $x^2 + y^2 + z^2 = a$ is the kernel of the linear map dg . This is the subspace spanned by

$$v_1 = \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \quad , \quad v_2 = \left(2xy \frac{\partial}{\partial z} - yz \frac{\partial}{\partial x} - xz \frac{\partial}{\partial y} \right)$$

The sets intersect transversally, if at every point of intersection the above subspaces span \mathbb{R}^3 . Since $u_1 = v_1$, this means that $u_2 \neq v_2$ and both non-zero. When $a > 1$, we have $z \neq 0$, and so this is indeed the case. So the intersection is transverse. When $a = 1$, we have $z = 0$, and so $u_2 = v_2$ and the intersection is non-transverse.

2. PROBLEM 9

At a point v in V , the derivative of the diagonal map induces

$$(D\Delta)_v(z) = (z, z) \in T_vV \oplus T_vV$$

The derivative of the graph of the linear map induces

$$(DW)_v(z) = (z, Az) \in T_vV \oplus T_{Av}V$$

Suppose 1 is not an eigenvalue of A . Then the only point of intersection for Δ and W is $(0, 0)$. Moreover, for any non-zero $z \in T_{(0,0)}(V \oplus V)$ the derivatives are unequal on the second factor.

Conversely, suppose that W is transverse to Δ . If there is a point of intersection (v, v) with $v \neq 0$ then $Av = v$. This implies that $(D\Delta)_v(v) = (DA)_v(v)$, so the intersection of the subspaces $(D\Delta)(T_vV)$ and $(DW)(T_vV)$ has dimension at least 1. By a dimension count the subspaces together cannot span $T_{(v,v)}(V \oplus V)$. So the only point of intersection is $(0, 0)$. By transversality, for all non-zero vectors z in $T_{(0,0)}V$ we must have $Az \neq z$. This is equivalent to saying that A does not have eigenvalue 1.

3. PROBLEM 10

(a): The map q factors through the map $Q : S^{n-1} \rightarrow \mathbb{R}P^{n-1}$, which is a two-fold cover of $\mathbb{R}P^{n-1}$. In particular, Q is proper and a local diffeomorphism. So given a vector $x \in V$ of norm 1, it is enough to show that in the local chart U of S^{n-1} containing x on which Q is a diffeomorphism, the set $S^{n-1} \cap V$ is a sub-manifold. This is obvious, and so we are done. In particular, if $L_x = \{w \in \mathbb{R}^n : \langle w, x \rangle = 0\}$ then using triviality of the $T\mathbb{R}^n$ i.e. $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ gives us L_x as the tangent space to S^{n-1} at x . The derivative dQ maps this diffeomorphically onto $T_{Q(x)}\mathbb{R}P^{n-1}$. If we restrict dQ to $V \cap L_x$ in T_xS^{n-1} then the image is $T_{q(x)}q(V)$ i.e. the tangent space at $q(x)$ to the sub-manifold $q(V)$.

(b): The tangent bundle to \mathbb{R}^n is trivial i.e. $T_x\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. We have $V \pitchfork W$ iff at every $x \in V \cap W$ we have $T_xV + T_xW = T_x\mathbb{R}^n$. Using triviality of the tangent bundle, this is equivalent to

$$V \pitchfork W \Leftrightarrow V + W = \mathbb{R}^n \Leftrightarrow \dim(V) + \dim(W) - \dim(V \cap W) = n$$

(c): By the reasons mentioned in (a), it is enough to check that $(S^{n-1} \cap V) \pitchfork (S^{n-1} \cap W)$.