Math 518 Differential Manifolds I

Take Home Exam, Due Friday November 6 by 5:00 PM
(in my mailbox or via email)

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1. Let M be a smooth manifold. Prove that the cotangent bundle of M,
$$T^*M = \bigcup_{x \in M} T_x^* M,$$
is also a smooth manifold.

2. (a) Prove that the subset $S \subset (\mathbb{R}^n)^l$ consisting of all linear independent l-tuples of vectors in $\mathbb{R}^n$ is open, and the map $\mathbb{R}^l \times S \to \mathbb{R}^n$ defined by
$$((t_1, \dots, t_l), (v_1, \dots, v_l)) \mapsto t_1 v_1 + \dots + t_l v_l$$
is a submersion.

   (b) Suppose that M is a compact submanifold of $\mathbb{R}^n$. Use part (a) to prove that "almost every" vector subspace V of $\mathbb{R}^n$ of dimension l, intersects M transversally. (You may quote the appropriate theorem we proved in class.)

3. Let f and F be a smooth functions on a compact manifold M, and let g be a Riemannian metric on M.

   (a) Let $\phi(t, x)$ be the flow of the vector field $F(\nabla_g f)$. Show that
$$\frac{d}{dt}(f(\phi_x(t)) = F(\phi_x(t)) \| \nabla_g f(\phi_x(t)) \|_g^2,$$
where $\| \nabla_g f(\phi_x(t)) \|_g^2$ is the value of $g(\nabla_g f, \nabla_g f)$ evaluated at $\phi_x(t)$.

   (b) Suppose that a and b are regular values of f with $a < b$, and that there are no critical values of f in the interval $[a, b]$. Prove that there is a diffeomorphism of M which maps the submanifold $f^{-1}(a)$ to $f^{-1}(b)$. Hint: use part (a).