Summary

1. Organizational issues

2. Systems of linear equations

1. Organizational issues

— First day handout

2. Systems of linear equations

Example 1.1 A plasma mixture contains protons and electrons. Total relative mass is 10.01 while the total relative charge is −10. Find the number of protons and electrons in the plasma mixture.

x_1 = number of protons
x_2 = number of electrons

$$(*)\begin{cases} x_1 \cdot 1 + x_2 \cdot 0.0005 = 10.01 \\ x_1 \cdot 1 - x_2 \cdot 1 = -10 \end{cases}$$

(*) is called a system of two linear equations with two unknowns. An equation is called linear if it made of a sum of the unknowns multiplied by coefficients.

Solution of (*):

Step 1 Eliminate x_1 from the second eq by adding to it the first equation multiplied by -1:

$$\begin{cases} x_1 + x_2 \cdot 0.0005 = 10.01 \\ x_1 - x_2 = -10 \end{cases} \iff \begin{cases} x_1 + x_2 \cdot 0.0005 = 10.01 \\ x_1 \cdot 0 - x_2 \cdot 1.0005 = -20.01 \end{cases}$$

Step 2 Solve for x_2 in the second eq by dividing by -1.0005:

$$\begin{cases} x_1 + x_2 \cdot 0.0005 = 10.01 \\ x_1 \cdot 0 - x_2 \cdot 1.0005 = -20.01 \end{cases} \iff \begin{cases} x_1 + x_2 \cdot 0.0005 = 10.01 \\ x_2 = 20 \end{cases}$$

Step 3 Eliminate x_2 from first equation by substituting its value or, equivalently by adding to the first eq the second one multiplied by -0.0005:

$$\begin{cases} x_1 + x_2 \cdot 0.0005 = 10.01 \\ \quad\quad x_2 = 20 \end{cases} \Leftrightarrow \begin{cases} x_1 \quad\quad = 10 \\ \quad\quad x_2 = 20 \end{cases}$$

The problem has one solution $x_1 = 10$, $x_2 = 20$.

The method called row reduction works on longer systems, with more unknowns and or equations:

<u>Example 2.1</u> A company produces three products and the costs associated with producing one unit of each are given in the following table

Product	1	2	3
Material cost/unit in \$	.45	.15	.30
Labor cost/unit in \$	.15	.35	.25
Overhead cost/unit in \$	.09	.13	.16

If the total costs in December were: materials \$ 90, labor \$ 75, overhead \$ 34, find out how many units of each product were made in December.

x_1 = number of units of product 1.
x_2 = number of units of product 2.
x_3 = number of units of product 3.

$$(**)\quad \begin{cases} x_1 \cdot 0.45 + x_2 \cdot 0.15 + x_3 \cdot 0.30 = 90 \\ x_1 \cdot 0.15 + x_2 \cdot 0.35 + x_3 \cdot 0.25 = 75 \\ x_1 \cdot 0.09 + x_2 \cdot 0.13 + x_3 \cdot 0.16 = 34 \end{cases}$$

A system of three linear eq with three unknowns.

Step 1 Eliminate x_1 from the second and third eq by adding to the second eq the first multiplied by $-\frac{1}{3}$ and adding to the third eq the first multiplied by $-\frac{1}{5}$.

$$\begin{cases} x_1 \cdot 0.45 + x_2 \cdot 0.15 + x_3 \cdot 0.30 = 90 \\ \qquad\qquad\quad x_2 \cdot 0.30 + x_3 \cdot 0.15 = 45 \\ \qquad\qquad\quad x_2 \cdot 0.10 + x_3 \cdot 0.10 = 16 \end{cases}$$

Step 2 Eliminate x_2 from the third equation by adding to it the second eq multiplied by $-\frac{1}{3}$

$$\begin{cases} x_1 \cdot 0.45 + x_2 \cdot 0.15 + x_3 \cdot 0.30 = 90 \\ x_2 \cdot 0.30 + x_3 \cdot 0.15 = 45 \\ x_3 \cdot 0.05 = 1 \end{cases}$$

<u>Step 3</u> Solve for x_3 in the third eq by dividing it by 0.05 :

$$\begin{cases} x_1 \cdot 0.45 + x_2 \cdot 0.15 + x_3 \cdot 0.30 = 90 \\ x_2 \cdot 0.30 + x_3 \cdot 0.15 = 45 \\ x_3 = 20 \end{cases}$$

<u>Step 4</u> Eliminate x_3 from Eq 1 and 2 by substituting, or equivalently, adding to Eq 2 the third one multiplied by -0.15 and adding to Eq 1 the third one multiplied by -0.30 :

$$\begin{cases} x_1 \cdot 0.45 + x_2 \cdot 0.15 = 84 \\ x_2 \cdot 0.30 = 42 \\ x_3 = 20 \end{cases}$$

Step 6 Solve for x_2 by dividing Eq 2 by 0.30

$$\begin{cases} x_1 \cdot 0.45 + x_2 \cdot 0.15 & = 84 \\ x_2 & = 140 \\ x_3 & = 20 \end{cases}$$

Step 7 Eliminate x_2 from Eq 1 by substituting or, equivalently by adding to Eq 1 the second one multiplied by -0.15

$$\begin{cases} x_1 \cdot 0.45 & = 63 \\ x_2 & = 140 \\ x_3 & = 20 \end{cases}$$

Step 8 Solve for x_1 by dividing Eq 1 by 0.45:

$$\begin{cases} x_1 & = 140 \\ x_2 & = 140 \\ x_3 & = 20 \end{cases}$$

The operations used to get the solution can be written in a condensed form using matrices:

The information in system (**) are contained in the following <u>extended matrix = table</u>

$$\begin{bmatrix} 0.45 & 0.15 & 0.30 & 90 \\ 0.15 & 0.35 & 0.25 & 75 \\ 0.09 & 0.13 & 0.16 & 34 \end{bmatrix}$$

Step 1–8 can be rewritten:

$$\begin{bmatrix} 0.45 & 0.15 & 0.30 & 90 \\ 0.15 & 0.35 & 0.25 & 75 \\ 0.09 & 0.13 & 0.16 & 34 \end{bmatrix} \sim \begin{bmatrix} 0.45 & 0.15 & 0.30 & 90 \\ 0 & 0.30 & 0.15 & 45 \\ 0 & 0.10 & 0.10 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0.45 & 0.15 & 0.30 & 90 \\ 0 & 0.30 & 0.15 & 45 \\ 0 & 0 & 0.05 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.45 & 0.15 & 0.30 & 90 \\ 0 & 0.30 & 0.15 & 45 \\ 0 & 0 & 1 & 20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0.45 & 0.15 & 0 & 84 \\ 0 & 0.30 & 0 & 42 \\ 0 & 0 & 1 & 20 \end{bmatrix} \sim \begin{bmatrix} 0.45 & 0.15 & 0 & 84 \\ 0 & 1 & 0 & 140 \\ 0 & 0 & 1 & 20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0.45 & 0 & 0 & 63 \\ 0 & 1 & 0 & 140 \\ 0 & 0 & 1 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 140 \\ 0 & 1 & 0 & 140 \\ 0 & 0 & 1 & 20 \end{bmatrix}$$

At each step (denoted by $\sim$) we did one or more operations of the type:

— add to one row another one multiplied by a number.

— multiply a row by a number

— swap two rows.

<u>Def</u> Two matrices that differ only by one or more operations of the above type are called row equivalent.

<u>Remark</u> If the extended matrices of two systems of linear equations are row equivalent then the two systems have the same solutions.

<u>Example 1.2</u> Like Example 1.1. but the mixture can contain x_3 neutrons:

$$x_1 \cdot 1 + x_2 \cdot 0.0005 + x_3 \cdot 1 = 10.01$$
$$x_1 \cdot 1 - x_2 \cdot 1 = -10$$

extended matrix: $\begin{bmatrix} 1 & 0.0005 & 1 & 10.01 \\ 1 & -1 & 0 & -10 \end{bmatrix}$

add to row 2 the row 1 multiplied by -1:

$$\begin{bmatrix} 1 & 0.0005 & 1 & 10.01 \\ 1 & -1 & 0 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.0005 & 1 & 10.01 \\ 0 & -1.0005 & -1 & -20.01 \end{bmatrix}$$

divide row 2 by -1.0005:

$$\begin{bmatrix} 1 & 0.0005 & 1 & 10.01 \\ 0 & -1.0005 & -1 & -20.01 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.0005 & 1 & 10.01 \\ 0 & 1 & 0.9999 & 20 \end{bmatrix}$$

add to row 1 the row 2 multiplied by -0.0005

$$\begin{bmatrix} 1 & 0.0005 & 1 & 10.01 \\ 0 & 1 & 0.9999 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.9995 & 10 \\ 0 & 1 & 0.9999 & 20 \end{bmatrix}$$

Rewrite the system of eq:

$$\begin{cases} x_1 + \qquad x_3 \cdot 0.9995 = 10 \\ \qquad x_2 + x_3 \cdot 0.9999 = 20 \end{cases} \Leftrightarrow \begin{cases} x_1 = 10 - 0.9995 \cdot x_3 \\ x_2 = 20 - x_3 \cdot 0.9999 \end{cases}$$

x_3 is free $\Rightarrow$ more than one solution.

Remark If x_3 could be any real number we would have infinitely many solutions. But since x_1, x_2, x_3 must be nonnegative integers in this case we only have 11 solutions corresponding to $x_3 = 0, 1, 2, 3, \ldots, 10$.

Example 2.2 Like example 2.1 but in January the company produces only the first two products and incurs the same costs:

$$\begin{cases} x_1 \cdot 0.45 + x_2 \cdot 0.15 = 90 \\ x_1 \cdot 0.15 + x_2 \cdot 0.35 = 75 \\ x_1 \cdot 0.09 + x_2 \cdot 0.13 = 34 \end{cases}$$

extended matrix : $\begin{bmatrix} 0.45 & 0.15 & 90 \\ 0.15 & 0.35 & 75 \\ 0.09 & 0.13 & 34 \end{bmatrix}$

$$\sim \begin{bmatrix} 0.45 & 0.15 & 90 \\ 0 & 0.30 & 45 \\ 0 & 0.10 & 16 \end{bmatrix} \sim \begin{bmatrix} 0.45 & 0.15 & 90 \\ 0 & 0.30 & 45 \\ 0 & 0 & 1 \end{bmatrix}$$

The last eq in the system is $x_1 \cdot 0 + x_2 \cdot 0 = 1$ impossible

The system has no solution. (Somebody made a mistake in the protocol data. For example the overhead costs are reported incorrectly and should have been $33 instead of $34.)

Conclusions :- System of linear equations can have a unique solution, no solutions or infinitely many solutions.

— Matrices are tables of numbers and can be used to represent linear systems of equations

— Row equivalent matrices correspond to systems with the same set of solutions

— Row reduction on matrices is used to bring systems to their simplest form and describe their solutions.