2.2 (continuation) The inverse of a matrix.

General properties of the inverse

(i) If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$

(ii) If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

(iii) If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Remark The above properties come directly from the definition of invertible matrices and properties of products of matrices. For example for (ii):

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

So AB is invertible with inverse $B^{-1}A^{-1}$.

Theorem Let A be an $n \times n$ matrix. A is invertible if and only if A is row equivalent to I_n. Moreover the inverse matrix can be calculated by row-reducing the augmented matrix $[A \; I_n]$:

$$[A \;\; I_n] \sim [I_n \;\; A^{-1}]$$

Example 1 Check whether

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$ is invertible. If yes find its inverse.

$$\left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{matrix}}_{A} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \end{array}\right] \overset{1^\circ}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{2^\circ}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \overset{3^\circ}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\overset{4^\circ}{\sim}\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \overset{5^\circ}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ \underbrace{0 \quad 0 \quad 1}_{I_3} & & & \underbrace{3/2 \quad -2 \quad 1/2}_{A^{-1}} & & \end{array}\right]$$

So $A \sim I_3 \Rightarrow A$ is invertible and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

Above we used the following row operations

1° Switch row 1 and 2
2° Subtract from row 3 four times row 1
3° Add to row 3 three times row 2
4° Divide row 3 by two
5° Subtract from row 2 two times row 3
and subtract from row 1 three times row 3.

Why does the algorithm work? To answer we need to understand 'elementary matrices':

Elementary matrices : These are all matrices that can be obtained from the identity matrix by performing one row operation.

Examples

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

E_{12} is obtained from I_3 by switching rows 1 and 2.

$E_{31}(4)$ is obtained from I_3 by adding to row 3 four times row 1.

Properties of elementary matrices

a) If E is a $n \times n$ elementary matrix and A is a $n \times m$ matrix then EA is the matrix which is obtained from A by doing the same row operation as the one done to obtain E from I_n.

<u>Examples</u> For A as in example 1

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

and E_{12} the elementary matrix obtained from switching the first two rows:

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$E_{12} A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix} \updownarrow \text{ first two rows are switched}$$

The steps performed in example 1 to transform A into I_3 can be rewritten as:

$$E_{13}(-3)\, E_{23}(-2)\, E_3(\tfrac{1}{3})\, E_{32}(3)\, E_{31}(-4)\, E_{12}\, A = $$

$$= I_3$$

Try to identify the elementary matrices used above!

b) Elementary matrices are invertible

This is because any row operation can be undone. For example to undo the switching of the first two rows you switch them again; so:

$$E_{12}\,E_{12} = I_n$$

To undo adding to row 3 four times row 1 you subtract from row 3 four times row 1:

$$E_{31}(-4)\,E_{31}(4) = I_n$$

Returning to <u>example 1</u> we have

$$\underbrace{\left(E_{13}(-3)\,E_{23}(-2)\,E_3\left(\tfrac{1}{3}\right)E_{32}(3)\,E_{31}(-4)\,E_{12}\right)}A = I_3$$

invertible as a product of invertible matrices

$$A = \underbrace{\left(E_{13}(-3)\,E_{23}(-2)\,E_3\left(\tfrac{1}{3}\right)E_{32}(3)\,E_{31}(-4)\,E_{12}\right)^{-1}}I_3$$

invertible as the inverse of an invertible matrix

So A is invertible and

$$A^{-1} = E_{13}(-3)\,E_{23}(-2)\,E_3(\tfrac{1}{3})\,E_{32}(3)\,E_{31}(-4)\,E_{12}\,I_3$$

which means that A^{-1} is obtained from I_3 by doing the same row operations that were performed to transform A into I_3

Remark. The above argument can be performed in general and shows that the theorem is true, see the textbook.