Lecture 12 Math 225

Note Title 2/25/2008

3.1. Determinants of 2×2, 3×3 and $n\times n$ matrices

__Def__ If $A = [a_{11}]$ then $\det A = a_{11}$.

If
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{then } \det A = a_{11}a_{22} - a_{21}a_{12}$$

__Remark__ In section 2.2 we established that $\det A$ "determines" whether A is invertible or not:

$$A \text{ invertible} \iff \det A \neq 0$$

__Def__ If
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then
$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

__Remark__ In the textbook you find a calculation showing that the above number should determine whether A is invertible or not :

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11} \det A \end{bmatrix}$$

<u>Methods of calculation of a 3×3 determinant</u>

I

$$\begin{matrix} & & & - & - & - \\ \end{matrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\begin{matrix} & & & + & + & + \\ \end{matrix}$$

<u>Example</u>

$$\det \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \begin{matrix} 0 & 1 \\ 3 & 4 \\ 6 & 7 \end{matrix}$$

$$= 0 \cdot 4 \cdot 8 + 1 \cdot 5 \cdot 6 + 2 \cdot 3 \cdot 7 - 6 \cdot 4 \cdot 2 - 7 \cdot 5 \cdot 0 - 8 \cdot 3 \cdot 1$$

$$= 30 + 42 - 48 - 24 = 0$$

II

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23})$$

$$- a_{12}(a_{21}a_{33} - a_{31}a_{23})$$

$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

where A_{ij} is obtained from A by removing row i and column j.

Example

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = 0 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix}$$

$$+ 2\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} = (-1)(-6) + 2(-3)$$

$$= 0$$

Def If $A = [a_{ij}]_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ $n \geq 2$ then

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{n+1} \det A_{1n}$$

where A_{ij} is obtained from A by removing row i and column j.

Remark This definition is a generalization of method II for 3×3 matrices. Method I does not generalize to larger matrices.

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{bmatrix} = 1 \cdot \begin{vmatrix} 5 & 0 & 0 \\ 6 & 8 & 0 \\ 7 & 9 & 10 \end{vmatrix}$$

$$= 1 \cdot 5 \begin{vmatrix} 8 & 0 \\ 9 & 10 \end{vmatrix} = 1 \cdot 5 \cdot 8 \cdot 10$$

$$= 400$$

Remark For an upper triangular matrix the determinant is the product of the main diagonal entries.

Def If $A = [a_{ij}]_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ $n \geq 2$

then

$$C_{ij} = (-1)^{i+j} A_{ij}$$

is called the co-factor of the entry (i,j).

Remark $\det A = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}$

Theorem If $A = [a_{ij}]_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ then

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

and

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$$

Example: By doing co-factor expansions down the 1^{st} column we have:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{vmatrix}$$

$$= 1 \cdot 5 \begin{vmatrix} 8 & 9 \\ 0 & 10 \end{vmatrix} = 1 \cdot 5 \cdot 8 \cdot 10$$

$$= 400$$

Remark The determinant of a lower triangular matrix is the product of the entries on the main diagonal.

Example:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 5 & 4 & 3 & 2 \\ 1 & 0 & -1 & -2 \end{vmatrix} = 5 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 2 \\ 1 & 0 & -2 \end{vmatrix}$$

$$+ 6 (-1)^{2+4} \begin{vmatrix} 1 & 2 & 3 \\ 5 & 4 & 3 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= -5\left[1\cdot(-1)^{3+1}\begin{vmatrix}2 & 4\\ 4 & 2\end{vmatrix} + (-2)(-1)^{3+3}\begin{vmatrix}1 & 2\\ 5 & 4\end{vmatrix}\right]$$

$$+ 6\left[1\cdot(-1)^{3+1}\begin{vmatrix}2 & 3\\ 4 & 3\end{vmatrix} + (-1)(-1)^{3+3}\begin{vmatrix}1 & 2\\ 5 & 4\end{vmatrix}\right]$$

$$= -5\left[(-12) + (-2)(-6)\right] + 6\left[(-6) + (-1)(-6)\right]$$

$$= 0 + 0 = 0$$