3.2 Properties of determinants

square matrixes

If A and B are row equivalent ; $A \sim B$
then their determinants are related as follows.

<u>Theorem</u> (Row operations) Let A be a square matrix.

a. If B is obtained from A by adding to a row of A another row multiplied by a number then

$$\det B = \det A$$

b. If B is obtained from A by interchanging two rows of A then

$$\det B = -\det A$$

c. If B is obtained from A by multiplying a row of A by k then

$$\det B = k \det A$$

Example

$$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -2 \\ -3 & -7 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 6 \\ 0 & 2 & 5 & -10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

$$= -6$$

Remark Row reduction brings a square matrix A to an upper triangular form U. Therefore

$$\det A = (-1)^r \det U$$
$$= (-1)^r \cdot \text{product of entries on the main diagonal of } U$$

where r is the number of row interchanges needed to bring A in the form U.

Corollary The following are equivalent

1° A is invertible

2° $\det A \neq 0$

This is because U in the previous remark has all entries on the main diagonal nonzero if A is invertible (why?) and at least one zero entry on the main diagonal if A is noninvertible (why?).

Proof of theorem c. is immediate by using cofactor expansion with respect to the row that is multiplied by k.

a. is similar to b.

b. Check for 2×2 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

$$\Rightarrow \det B = a_{21} \cdot a_{12} - a_{11} a_{22} = -(a_{11} a_{22} - a_{21} a_{12})$$
$$= -\det A$$

So the property is valid for 2×2 matrices

<u>Claim</u> Property valid for 2×2 matrices implies the property is valid for 3×3 matrices

Indeed use a cofactor expansion with respect to a row unaffected by the interchange:

$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + a_{i3} C_{i3}$

$\det B = a_{i1} \tilde{C}_{i1} + a_{i2} \tilde{C}_{i2} + a_{i3} \tilde{C}_{i3}$

the cofactors are determinants of 2×2 matrices and $\tilde{C}_{ij}$ differs by C_{ij} by a row interchange
$\Rightarrow$ $\tilde{C}_{ij} = - C_{ij}$ so

$\det B = a_{i1} (-C_{i1}) + a_{i2} (-C_{i2}) + a_{i3} (-C_{i3})$

$= - \det A.$

Similarly from the property valid for 3×3 matrices we can get that the property is valid for 4×4 matrices. Using <u>induction</u> the property is valid for any square matrix.

<u>Theorem</u> If A is a square matrix then $\det(A^T) = \det A$.

<u>Proof</u> Check for 2×2 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$\det(A^T) = a_{11}a_{22} - a_{12}a_{21} = \det A$$

For 3×3 and larger matrices use induction as in the proof of b. for the previous theorem

<u>Corollary</u> (column operations)

a. If B is obtained from A by adding to a column of A another column of A multiplied by a number then $\det B = \det A$

b. If B is obtained from A by interchanging two columns of A then $\det B = -\det A$

c. If B is obtained from A by multiplying a column of A by k then $\det B = k \det A$.

Proof : b. Assume B is obtained from A by interchanging columns j_1 and j_2.
$\Rightarrow$ B^T is obtained from A^T by interchanging rows j_1 and j_2.

$$\Rightarrow \det B^T = -\det A^T$$
$$\Rightarrow \det B = \det B^T = -\det A^T = -\det A.$$

a. and c. are shown similarly.

Example

$$\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 0$$

Subtract from column 3 twice column 1.

Theorem (product of matrices and determinants)
If A and B are square matrices then

$$\det(AB) = (\det A)(\det B)$$

<u>Application</u> If A invertible then:

$$(AA^{-1}) = I_n \Rightarrow \det(AA^{-1}) = 1$$

$$\Rightarrow (\det A)(\det A^{-1}) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det A}$$