Lecture 14 Math 225

Note Title 3/1/2008

3.3 Cramer's Rule and an inverse formula

Notation If A has the columns given by vectors $a_1, a_2, \dots, a_n$ in $\mathbb{R}^n$:

$$A = [\, a_1 \ \ a_2 \ \dots \ a_j \dots \ a_n \,]$$

then $A_j(b)$ is the matrix obtained from A by replacing its j^{th} column by b in $\mathbb{R}^n$

$$A_j(b) = [\, a_1 \ a_2 \dots \underset{\uparrow j^{th} \text{ column}}{b} \dots a_n \,]$$

Theorem (Cramer's Rule) If A is an $n \times n$ invertible matrix then the unique sln of

$$Ax = b$$

is given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{with } x_i = \frac{\det(A_i(b))}{\det A}$$

Application 1 :

$$\begin{cases} x_1 + 2x_2 = 3 \\ 4x_1 + 5x_2 = 6 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} ;$$

$A \nearrow$

$$\det A = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$$

So Cramer Rule applies (why?)

↙ RHS becomes first column

↙ Second column unchanged

$$x_1 = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{-3} = \frac{15-12}{-3} = \underline{-1}$$

↙ first column unchanged

← RHS becomes second column

$$x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{-3} = \frac{6-12}{-3} = \frac{-6}{-3} = 2$$

Proof of the theorem: let I be the identity matrix: $I = [e_1\ e_2 \dots e_i \dots e_n]$

$$e_i = \begin{bmatrix} 0 & \\ 0 & \\ \vdots & \\ 1 & \leftarrow \text{row } i \\ \vdots & \\ 0 & \end{bmatrix}$$

$$I_i(x) = [\,e_1\ e_2 \dots \overset{\text{column } i}{\overset{\swarrow}{x}} \dots e_n\,]$$

$$= \begin{array}{r c} & \left[\begin{array}{cccccc} 1 & 0 & 0 & \dots & x_1 & \dots & 0 \\ 0 & 1 & 0 & \dots & x_2 & \dots & 0 \\ 0 & 0 & 1 & \dots & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & x_i & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & x_n & \dots & 1 \end{array}\right] \end{array}$$

row i $\rightarrow$ (row of x_i)

Cofactor expansion with respect to row j for:

$$\det I_i(x) = x_i\ (-1) \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix} = x_i$$

$$A I_i(x) = [\, A e_1 \;\; A e_2 \; \dots \; \overset{\text{column } i}{Ax} \; \dots \; A e_n \,].$$

$$= [\, a_1 \;\; a_2 \; \dots \; \underset{= b}{Ax} \; \dots \; a_n \,]$$

$$= A_i(b)$$

So

$$\det A_i(b) = \det [A I_i(x)] = \det A \cdot \det I_i(x)$$

$$= \det A \cdot x_i$$

Hence $x_i = \dfrac{\det A_i(b)}{\det A}$ QED.

Application 2 : The method of variation of constants says that the solution of the following second order differential equation related to oscillatory systems :

$$x''(t) + x(t) = f(t)$$

is given by

$$x(t) = C_1(t) \cos t + C_2(t) \sin t$$

where :

$$C_1'(t)\cos t + C_2'(t)\sin t = 0$$
$$C_1'(t)(-\sin t) + C_2'(t)\cos t = f(t)$$

Via Cramer rule we get:

$$C_1'(t) = \frac{\begin{vmatrix} 0 & \sin t \\ f(t) & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = -f(t)\sin t$$

$$C_2'(t) = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & f(t) \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = f(t)\cos t$$

If the forcing function $f(t)$ is known one gets:

$$x(t) = \left[-\int f(s)\sin s\, ds + C_1\right]\cos t + \left[\int f(s)\cos s\, ds + C_2\right]\sin t$$

where C_1, C_2 are arbitrary constants that can be determined from the initial conditions $x(0)$ and $x'(0)$.

Theorem (an inverse formula)
An $n \times n$ matrix A is invertible if (and only if) $\det A \neq 0$ in which case:

$$(*) \quad A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

where C_{ij} is the cofactor of the (i,j) entry.

Proof let $A^{-1} = [b_1 \; b_2 \; \ldots \; b_n]$ and the identity: $I = [e_1 \; e_2 \; \ldots \; e_n]$

$AA^{-1} = I$ becomes.

$$[Ab_1 \; Ab_2 \; \ldots \; Ab_n] = [e_1 \; e_2 \ldots e_n]$$

So

$Ab_j = e_j$

The cramer rule gives us the i^{th} entry of b_j, or the (i,j) entry of A^{-1}:

$$b_{ij} = \frac{\det A_i(e_j)}{\det A}$$

$$A_i(e_j) = \begin{bmatrix} a_{11} & a_{12} & \cdots & \overset{\text{column } i}{0} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{j1} & a_{j2} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & a_{nn} \end{bmatrix} \leftarrow \text{row } j$$

Cofactor expansion w.r.t column i gives

$$\det A_i(e_j) = 1 \cdot C_{ji}$$

So $b_{ij} = \dfrac{C_{ji}}{\det A}$ and $A^{-1} = [b_{ij}]_{\substack{1 \le i \le n \\ 1 \le j \le n}}$

gives formula $(*)$. QED

<u>Application</u> Check whether

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$ is invertible. If it

is compute its inverse.

$$\det A = \begin{vmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{vmatrix} = 3 \cdot (-1)^{2+1} \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= -3(-2+1) = 3 \neq 0$$

So A is invertible and.

$$A = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \qquad C_{21} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{12} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3 \qquad C_{22} = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1$$

$$C_{13} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3 \qquad C_{23} = -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} = 0$$

$$C_{32} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3, \quad C_{33} = \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} = 6$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}$$