Lecture 16 Math 225

Note Title 3/13/2008

Sect 4.1 Vector spaces and vector subspaces

Def A real vector space V is a collection of objects on which two operations have been defined: for u, v in V and c real number $u, v \longmapsto u+v$ in V and $c, u \longmapsto cu$ in V. The operations must satisfy the following eight properties (axioms):

1. $u+v = v+u$ for any u, v in V;
2. $(u+v)+w = u+(v+w)$ for any u, v, w in V;
3. there exist 0 in V such that
   $0+u = u+0 = u$ for any u in V
4. for any u in V there exists $-u$ in V such that $(-u)+u = u+(-u) = 0$;
5. $(cd)u = c(du)$ for any c, d in $\mathbb{R}$ and u in V;
6. $(c+d)u = cu+du$ for any c, d in $\mathbb{R}$ and u in V;
7. $c(u+v) = cu+cv$ for any c in $\mathbb{R}$ and u, v in V;
8. $\mathbb{1} \cdot u = u$ for any u in V.

Examples $\mathbb{R}^n$ (for $n = 1, 2, 3, \dots$), the collection of all polynomials with the standard addition and multiplication by a constant.

- The collection of all real valued functions defined on an interval with the standard addition of two functions and multiplication of a function by a constant.

<u>Remark</u> We will see that based solely on the eight properties in the definition, vector spaces have rich properties. These properties are common to all vector spaces and that is what makes vector spaces a powerfull tool in science.

For example for any v in V we have

9. $\underset{\text{number}}{0} \cdot v = \underset{\text{vector}}{\mathbb{O}}$

10. $(-1) \cdot v = -v$

9. follows from:

$$0v = (0+0)v \overset{6.}{=} 0v + 0v$$

adding $-0v$ which exists by 3. we get

$$\mathbb{O} = 0v$$

10. follows from

$$0 \overset{9.}{=} 0v = (1+(-1))v \overset{6.}{=} v + (-1)v$$

adding $(-v)$ we get

$$-v = (-v) + 0 = ((-v)+v) + (-1)v = (-1)v$$

<u>Def</u> V_1 is a subspace of a real vector space V if and only if V_1 is a nonempty subset of V and:

(i) for any v, v in V_1 we have $v + v$ is in V_1

(ii) for any v in V_1 and c in $\mathbb{R}$ we have cv is in V_1

<u>Examples</u> A line or a plane are subspaces of $\mathbb{R}^3$, polynomials of degree at most 3 form a subspace of the vector space of all polynomials, polynomials are a subspace of the vector space of all real valued functions defined on the real line.

Remark Because a subspace V_1 is closed under addition and multiplication ((i) and (ii) in the definition) it becomes itself a real vector space. In other words V_1 satisfies 1.-8. In practice, to show that V_1 is a vector space one can replace checking 1.-8. with identifying V_1 with a subset of a known vector space V and checking (i)-(ii).

For example:

— We know the collection of all polynomials is a subset of the collection of all real valued functions of real variable

— we know that the real valued functions of real variable form a vector space

— We ~~check~~:

- Sum of two polynomials is a polynomial
- a number times a polynomial is a polynomial

— We deduce that polynomials form a vector space.

Problem Check whether the set V_1 of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ with $a+2b+3c=0$ is a subspace of $\mathbb{R}^3$

Check (i) Let $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ be in V_1

then $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix}$

properties of real #

and $(a_1+a_2)+2(b_1+b_2)+3(c_1+c_2) = a_1+2b_1+3c_1 + a_2+2b_2+3c_2 = 0+0 = 0.$

$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ are in V_1

So $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is in V_1 and (i) check.

Check (ii) Let d in $\mathbb{R}$ and $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in V_1 then

$d \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} da \\ db \\ dc \end{bmatrix}$ and

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$da + 2db + 3dc = d(a+2b+3c) = d \cdot 0 = 0$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in V_1

so $d \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in V_1 and (ii) checks