Lecture 17 Math 225

Note Title 3/13/2008

4.1 (Continuation) Linear combination and spans
4.2 The Null space and Column space of a matrix

4.1. Linear combination and spans

<u>Def</u> If V is a vector space and $v_1, v_2, \dots, v_p$ are in V while $c_1, c_2, \dots, c_p$ are real numbers then the vector:

$$(\dots((c_1 v_1 + c_2 v_2) + c_3 v_3) + \dots + c_{p-1} v_{p-1}) + c_p v_p$$

is called the linear combination of $v_1, v_2, \dots, v_p$ with weights $c_1, c_2, \dots, c_p$.

<u>Remark</u> The parentheses in the formula above are not necessary because of property 2. of vector spaces which allows us to group the terms of the sum however we want. From now on the linear combination will be denoted:

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

Def Let V be a vector space and $v_1, v_2, \dots, v_p$ in V then

$$\text{Span}\{v_1, v_2, \dots, v_p\}$$

is the collection of all linear combinations of $v_1, v_2, \dots, v_p$.

Theorem Let V be a vector space and $v_1, v_2, \dots, v_p$ in V. Then $\text{span}\{v_1, v_2, \dots, v_p\}$ is a subspace of V.

Proof Check that $\text{span}\{v_1, v_2, \dots, v_p\}$ is closed under addition:

Let u_1, u_2 in $\text{span}\{v_1, v_2, \dots, v_p\}$ then there are weights $c_1, c_2, \dots, c_p$ and $d_1, d_2, \dots, d_p$ such that

$$u_1 = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$
$$u_2 = d_1 v_1 + d_2 v_2 + \dots + d_p v_p$$

So

$$u_1 + u_2 = (c_1 v_1 + c_2 v_2 + \dots + c_p v_p) + d_1 v_1 + d_2 v_2 + \dots + d_p v_p$$

Change the order (allowed by 1.) and regroup (by 2.) until:

$$u_1 + u_2 = (c_1 v_1 + d_1 v_1) + (c_2 v_2 + d_2 v_2) + \dots + (c_p v_p + d_p v_p)$$

Use property 6.

$$u_1 + u_2 = (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \dots + (c_p + d_p) v_p$$

Hence v_1+v_2 is a linear combination of $v_1, v_2, \ldots, v_p$ with weights $c_1+d_1, c_2+d_2, \ldots, c_p+d_p$ and (i) in definition of subspace (see lecture 16) checks.

Check that $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is closed under multiplication by scalars:

Let c in $\mathbb{R}$ and v in $\text{span}\{v_1, v_2, \ldots, v_p\}$ then there are weights $c_1, c_2, \ldots, c_p$ such that

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_p v_p.$$

Now

$$\begin{aligned} cv &= c(c_1 v_1 + c_2 v_2 + \ldots + c_p v_p) \overset{7.}{=} \\ &= c(c_1 v_1) + c(c_2 v_2) + \ldots + c(c_p v_p) \overset{5.}{=} \\ &= (cc_1) v_1 + (cc_2) v_2 + \ldots + (cc_p) v_p \end{aligned}$$

So cv is a linear combination of $v_1, v_2, \ldots, v_p$ with weights $cc_1, cc_2, \ldots, cc_p$ and (ii) in definition of a subspace checks. QED

Examples the origin, lines through the origin and planes through the origin are all subspaces of $\mathbb{R}^3$. In fact they are the only subspaces of $\mathbb{R}^3$ except $\mathbb{R}^3$.

4.2 The null space and column space of a matrix

Let $A = [a_1 \ a_2 \ \dots \ a_n]$ be a $m \times n$ matrix

Def Null A is the set of all vectors x in $\mathbb{R}^n$ for which $Ax = 0$.

Example $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is not in Null A because

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ is in Null A because

$$A \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can find explicitly Null A by solving

$$Ax = 0$$

Use row reduction for $[A\ \ 0]$ and write the solution in the parametric vector form.
For the example above we have

$$[A\ \ 0] \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_3 is free

$\Rightarrow x_4 = 0$, x_3 is free, $x_2 = -2x_3$, $x_1 = -3x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

<u>Theorem</u> If A is an $m \times n$ matrix then Null A is a subspace of $\mathbb{R}^n$.

<u>Proof</u> If x_1, x_2 are in Null A then
$A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0$ so $x_1 + x_2$ is in Null A, (i) in definition of subspace checks.

If x_1 is in Null A and c is in $\mathbb{R}$ then

$$A(c x_1) = c\, A x_1 = c \cdot 0 = 0.$$

So $c x_1$ is in Null A and (ii) in the definition of a subspace check too. QED.

Remark The column of A are linearly independent if and only if Null $A = \{0\}$. See section 1.7 in your textbook

Def If $A = [a_1\ a_2\ \dots\ a_n]$ is an $m \times n$ matrix then

$$\text{col } A = \text{Span}\{a_1, a_2, \dots, a_n\}$$

Remark col A is a subspace of $\mathbb{R}^m$

It is easy to find vectors in col A. For example

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{then } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is in col } A \text{ or}$$

$$2 \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} \text{ is in col } A.$$

To check whether a given vector $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in col A one must find x in $\mathbb{R}^n$ such that

$$Ax = b.$$

In our example

$$[A \,.\, b] \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

↑ non pivot column

$\Rightarrow Ax = b$ has at least one sln $\Rightarrow b$ is in col A.

<u>Remark</u> col $A = \mathbb{R}^m$ if and only if A has a pivot position in each row. See section 1.4 in your textbook