Lecture 18 Math 225

Note Title 3/24/2008

Summary
- Short Review of Sect 4.1, 4.2.
- Sect 4.3

Short Review of Sect 4.1, 4.2:

Q1. What is a (real) vector space? Ex?

Q2. What is a subspace of a vector space? Examples?

Q3. What is a linear combination of vectors? What is the span of a set of vectors?

Q4. Given the matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \end{bmatrix}$$

What is col A, Null A? Is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ in Null A? Is $\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$ in col A?

Sect 4.3: Linearly independent sets. Bases

Def (Linear independence) The set of vectors $\{v_1, v_2, \ldots, v_p\}$ in vector space V is called linearly independent if the vector equation:

$$(1) \qquad c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0 \qquad c_1, c_2, \ldots, c_p \text{ in } \mathbb{R}$$

has only the solution $c_1 = 0, c_2 = 0, \ldots, c_p = 0$.

Examples: $\{\sin t, \cos t\}$ in the vector space of all functions are linearly independent.

$\{1, t, t^2\}$ in $\mathbb{P}$ = vector space of all polynomials are linearly independent.

Def (linear dependence) The set of vectors $\{v_1, v_2, \ldots, v_p\}$ in the vector space V is called linearly dependent if there exist weights $c_1, c_2, \ldots, c_p$ in $\mathbb{R}$ not all zero such that

$$(2) \qquad c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0$$

Examples $\{t, 2t\}$; $\{\sin^2 t, \cos^2 t, 1\}$.

Remark If 0 is among $\{v_1, v_2, \dots, v_p\}$ then the set is linearly dependent

Theorem The set $\{v_1, v_2, \dots, v_p\}$ with $p \geq 2$ and $v_1 \neq 0$ is linearly dependent if and only if some v_j $(j > 1)$ is a linear combination of the preceding vectors.

Proof the "if" part requires to show that:

if $v_j = c_1 v_1 + c_2 v_2 + \dots + c_{j-1} v_{j-1}$ $c_1, c_2, \dots, c_{j-1}$ in $\mathbb{R}$
then $\{v_1, v_2, \dots, v_p\}$ are linearly dependent

Indeed: add $-v_j = (-1) v_j$ to the identity above

$$v_j + (-v_j) = c_1 v_1 + c_2 v_2 + \dots + c_{j-1} v_{j-1} + (-1) v_j$$

$$\Leftrightarrow \quad 0 = c_1 v_1 + c_2 v_2 + \dots + c_{j-1} v_{j-1} + (-1) v_j + 0 \cdot v_{j+1} + 0 \cdot v_{j+2} + \dots + 0 \cdot v_p$$

So there exist weights $c_1, c_2, \dots, c_{j-1}, (-1), 0, 0, \dots, 0$ not all zero such that (2) holds. Hence $\{v_1, v_2, \dots, v_p\}$ are linearly dependend

the "only if" part requires that $\{v_1, v_2, \dots, v_p\}$ linearly dependent and $v_1 \neq 0$ implies the existence of a v_j ($j>1$) and weights $c_1, c_2, \dots, c_{j-1}$ such that

$$v_j = c_1 v_1 + c_2 v_2 + \dots + c_{j-1} v_{j-1}$$

Indeed: there exists weights $c_1, c_2, \dots, c_p$ <u>not all zero</u> such that

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0$$

Choose the largest index j such that $c_j \neq 0$. Then

$$c_1 v_1 + c_2 v_2 + \dots + c_j v_j = 0 \quad (\text{add } -c_j v_j)$$

$$\Leftrightarrow \quad c_1 v_1 + c_2 v_2 + \dots + c_{j-1} v_{j-1} = (-c_j) v_j \quad (\text{divide by } (-c_j))$$

$$\Leftrightarrow \quad \left(-\frac{c_1}{c_j}\right) v_1 + \left(-\frac{c_2}{c_j}\right) v_2 + \dots + \left(\frac{-c_{j-1}}{c_j}\right) v_{j-1} = v_j .$$

<u>Def</u> (bases) Let H be a subspace of V. The set $\mathcal{B} = \{b_1, b_2, \dots, b_p\}$ is a <u>basis</u> for H if

(i) $\mathcal{B}$ is a linearly independent set

(ii) $H = \operatorname{span}\{v_1, v_2, \dots v_p\}$.

<u>Examples</u> $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

$\{1, t, t^2\}$ is a basis for $\mathbb{P}_2$ = the subspace of of polinomials of degree less or equal to 2.

Exercise Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$? If not can you obtain one by deleting some vectors from the set?

Answer: It is not a basis because it is not a linearly independent set:

$$\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \end{bmatrix}$$

↑ non pivot column.

However $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

because

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

- has only pivot columns => its columns are linearly independent
- has a pivot position in each row => its columns span $\mathbb{R}^3$

<u>Theorem</u> (The spanning set theorem)

Let $H = \text{Span}\{v_1, v_2, \ldots, v_p\}$ be a subspace of V. Denote $S = \{v_1, v_2, \ldots, v_p\}$.

a. If v_k is a linear combination of the other vectors in S then by removing it we obtain a set that still spans H.

b. If $H \neq \{0\}$ then a subset of S is a basis for H.

<u>Basis for Col A and Nul A</u>

$$\text{Let } A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & 7 \end{bmatrix}$$

To find a basis for col A use the echelon form of A:

$$A \sim B = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Then 1, 2 and 4^{th} column of A (NOT of B = echelon form of A) is a basis for col A.

$$\text{col } A = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 3 \\ -9 \end{bmatrix} \right\}$$

To find a basis for Nul A use the reduced echelon form of A

$$A \sim C = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

write the solutions of $Ax = 0$ in the vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_3 - 5x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ 0 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.